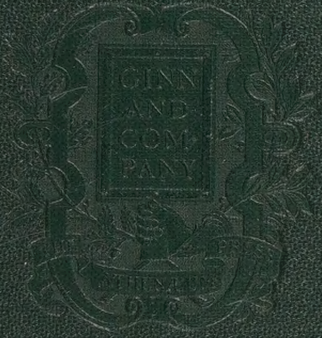
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ENGINEERING SERIES

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DESCRIPTIVE GEOMETRY

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PREFACE

Descriptive geometry deals with the fundamental theories underlying the only universal language; namely, the graphical language, or drawing. As the name implies, the descriptions must be in the form of geometrical constructions, comprehending both plane-geometry and solid-geometry materials and methods. Its function is to describe, in geometrical terms, the shape, size, space location, and space relations of geometrical magnitudes.

Numerous textbooks on descriptive geometry have been written since Professor A. E. Church published his memorable work on the subject in 1864. Each one has had some special point of interest and worth to recommend it to both teacher and student. Some have produced new methods of attack on old problems; others have offered new problems for solution with old methods. There have been many attempts during the last half century to make descriptive geometry clearer and easier to comprehend by textbook presentations, but it is quite evident to the impartial observer that the authors of these texts have taken the occasion to emphasize a certain method of attack, to the neglect, and sometimes the complete exclusion, of other equally good methods.

On account of the character of the subject matter of descriptive geometry and its universal application in both pure and applied science and art, it would seem beyond question that those principles and methods found to be either necessary or useful to a thorough understanding and application of the theory involved should not be omitted or abridged in the preparation of any treatise on the subject. Nevertheless many textbooks fail to relate or harmonize in their early chapters the several equally good methods of approach to the subject.

In presenting this text, the authors lay claim to no new discovery in the realm of mathematical science. They suggest no workless road to the mastery of a difficult subject. They find it impossible to agree in principle or in practice with any effort to restrict in methods and materials a wholly fundamental and general field of mathematical science. Their intention has been to prepare a fundamental treatise on descriptive geometry, and, in so doing, they have kept uppermost in mind the importance of weighing impartially the merits of each principle and method and of so arranging, relating, and emphasizing those selected that the student and teacher alike shall find the subject matter easier to comprehend and more readily applicable to the problems of our industrial and scientific life.

It is recognized that there are many specialized points of view from which one may proceed in the exposition of principles and in the choice of illustrations. The architect, the machine or structural designer, the geologist, the topographer, the crystallographer, and others use certain definite small portions of the descriptive geometry in their professional work. The authors have avoided overemphasis in any of the fields of application of theory to practical drafting. Selection of illustrations from any source has been made for the single purpose of properly illustrating the principle involved. The fundamental and general character of the subject has been the only basis upon which they have been willing to build.

So far as is known, no previous attempt has been made to accomplish the logical task of correlating the two primary and well-determined methods of solution of geometrical problems as each underlying principle is presented in the text. The authors have found this scheme eminently successful in their classes over a goodly period of years, principally because there is in the student's mind, as he progresses from the simple to the intricate, no sense of clashing between the theoretical and the practical. The auxiliary-plane method is used to supplement and clarify the older and more formal methods, and vice versa. This is one of the prime functions of the text and, we hope, the one it most successfully accomplishes.

The gradual elimination of second-quadrant and fourth-quadrant constructions as the topics progress toward those dealing with surfaces and solids is a feature of no small importance. The Analyses have been restricted to very general statements of fact, which lead to the final conclusion or solution of the problem. On the other hand, the Constructions have been elaborated to explain both the sequence and the meaning of each step as it is taken. Discussions have been inserted at appropriate places to bring out related facts and conclusions.

The authors take the opportunity to express their appreciation of the many helpful suggestions made by members of the Department of General Engineering Drawing at the University of Illinois during the preparation of the text, and especially their indebtedness to Professor R. A. Hall, who has read the text manuscript and offered many constructive criticisms of it.

H. H. JORDAN
F. M. PORTER

URBANA, ILLINOIS

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DESCRIPTIVE GEOMETRY

CHAPTER I

PROJECTIONS OF POINTS

1. Definition

Descriptive geometry, as its name implies, deals with the problems of shape description and space location of geometrical or physical magnitudes. It is concerned with the solutions of problems that arise in connection with the determination of distances, angles, intersections, and other relations between geometrical magnitudes. The language in which these shape descriptions are written is the graphical language, that is, free-hand and mechanical drawing. The magnitudes dealt with range from the infinitesimal point on through the whole catalogue of lines, planes, surfaces, and solids, to the most complex machines and structures.

The prime function of descriptive geometry is to develop principles and methods which will permit us to describe a magnitude and to relate it to other magnitudes, or to relate the several parts of a magnitude to each other, so that ideas and designs may be worked out and transmitted from one person to another by a simple but effective means called drawings. A thorough understanding of the methods and devices employed in the development of the subject will comprehend such things as coördinate planes, coördinates, orthographic projection, projectors, points of sight, views, and the like. Many principles of plane and solid geometry are foundation stones in the descriptive-geometry structure and must be drawn upon freely in solving stated problems.

2. Coördinate planes

Drawings are usually made on planes that are set up perpendicular to each other; for example, a horizontal and a vertical plane. In descriptive geometry three mutually perpendicular planes are sufficient for most purposes, but additional planes perpendicular to one or another of these first three may be set up for special reasons. The three primary planes are called the *principal* coördinate planes or simply *the* coördinate planes. They constitute the same set of planes found in other coördinate geometries, and their intersections the same coördinate axes, X, Y, and Z. These axial lines are called ground lines in descriptive geometry. The additional planes, referred to above, are called *auxiliary* coördinate planes, meaning, of course, helping planes. They are to be used when solutions on the principal planes become involved and laborious, or when the shape of an object is brought out more clearly by views drawn upon them than by the primary views.

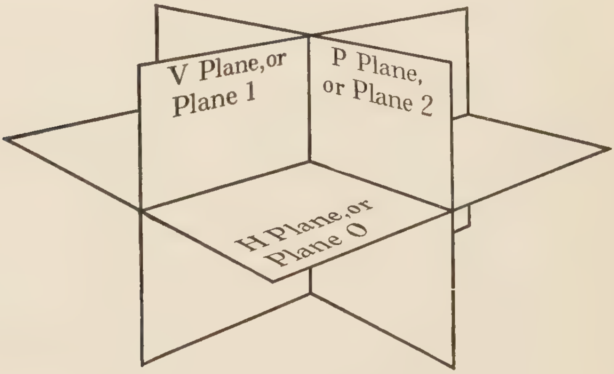


FIG. 1. The principal coördinate planes

One of the three primary coördinate planes is always assumed horizontal, that is, parallel to the plane determined by the spirit level. It may be thought of as the floor of a room or the top of a desk, for visualization purposes. This plane is designated the horizontal plane or simply the H plane. Another of the primary coördinate planes is set up perpendicular to the horizontal plane. It may be thought of as the front wall of a room and is designated the vertical plane or simply the V plane. The third coördinate plane is placed perpendicular to the H and V planes and is called the profile plane or simply the P plane. It may be thought of as either side wall of a room, but more often as on the right of the observer as he faces the V plane. (See Fig. 1.)

Each of the coördinate planes is infinite in extent. Only small portions of them are shown in Fig. 1. They are assumed to be transparent, although in actual work they are nothing more or less than the drawing paper, blackboard, or other materials upon which drawings are made. In practice, either the coördinate planes are assumed in some fixed position in space and the magnitudes being dealt with are placed within their included angles, or the magnitudes are assumed fixed in position and the planes placed about them. Fig. 2 shows an object placed in one chosen position in space relative to the coördinate planes.

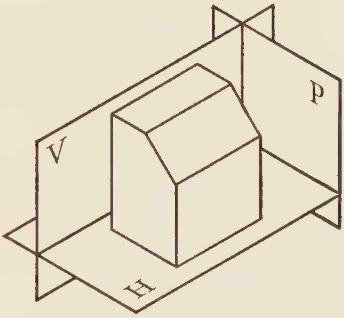


FIG. 2. Object and coördinate planes

3. Designation of coördinate planes

For convenience in marking drawings or projections, each coördinate plane is given a numerical designation in addition to the descriptive terms used in a preceding paragraph. Beginning with the H plane, they are marked and referred to in the following order: the H plane is called Plane 0, the V plane is called Plane 1, and the P plane is called Plane 2 (see Fig. 1). When the first auxiliary coördinate plane is set up in any problem, it is called Plane 3. If an auxiliary plane is set up perpendicular to Plane 3, it is called Plane 4. An auxiliary plane perpendicular to number 4 will be called Plane 5, and so on for as many coördinate planes as may be set up in any given problem.

4. Dihedral angles or quadrants

Considering only the horizontal and vertical planes, these two divide all space into four parts called angles or quadrants. That portion of space in front of the vertical plane and above the horizontal plane is called the first angle or quadrant; that portion of space behind the V plane and above the H plane is called the second angle or quadrant; that portion of space behind the V plane and below the H plane, the third angle or quadrant; and that in front of the V plane and below the H plane, the fourth angle or quadrant. The term "quadrant" is preferred to "angle" and will be used generally throughout this text. It should be noted that the profile plane has nothing to do with the quadrant designations, and further, that the term "quadrant" is used in the sense of a space designation rather than as the arc of a circle in a plane. Fig. 3 will assist in visualizing these definitions.

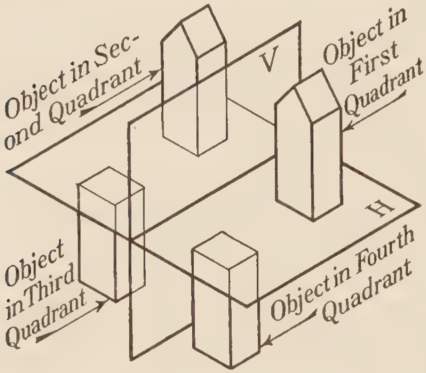


FIG. 3. Quadrant numbering

5. Projectors and projections

As already indicated, one of the purposes of setting up the coördinate planes is that of representing upon them all kinds of magnitudes or objects in such a way that complete information about the things dealt with may be given without the use of long written descriptions. The method employed in obtaining these representations is that of projection, or projecting, as it is called.

If an object, such as that shown in Fig. 4, is placed in any quadrant, and lines perpendicular to each coördinate plane are drawn through the significant corners of the object, the first step in the procedure of obtaining projections has been taken. These perpendicular lines, as Aa', Dd, Ff_2, etc., are called the *projectors.* Now, if the projectors are extended until they pierce the coördinate plane to which they are perpendicular, and the several piercing points in a plane are joined in the proper order, there results a drawing on each plane that is the exact size and shape of any face of the object to which a particular coördinate plane is parallel. Faces not parallel to any coördinate plane will be represented but not, of course, in true size. These drawings are called *projections* or *views* and, since the projectors to any one coördinate plane are all parallel to each other and perpendicular to that plane, this type of projection is called *orthographic* projection.

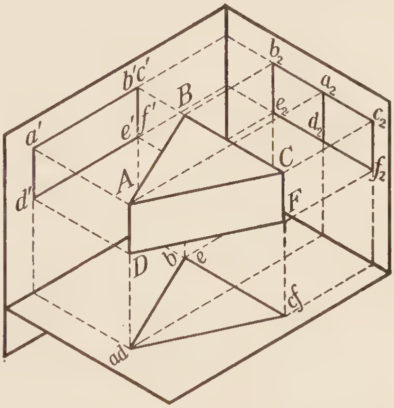


FIG. 4. Projections or views of an object

An orthographic projection, then, is a representation or view of any geometrical magnitude on a coördinate plane by means of drawings (projections) which result from connecting the piercing points, in the plane, of a number of parallel projectors drawn through the significant points of the magnitude and perpendicular to the plane. A number of these orthographic projections of an object, when taken together, will accurately and completely show the shape and size of the object represented. More than two coördinate planes are often necessary to secure these desirable results and sometimes as many as four or five planes are used. On account of this use of the coördinate planes in projective work, they are called *planes of projection.*

6. Marking projections; nomenclature

In Fig. 4 it was seen that the corner A of the wedge was projected upon three planes of projection. It is necessary that each of these representations of the point A have a characteristic marking to distinguish it from corresponding projections of other points, such as B and C, for instance. All projections of a point are marked with a small letter corresponding to the capital letter used to designate the point in space. A numerical subscript is then added to each small letter to designate the number of the coördinate plane upon which the projection is found. Thus, in the case of the H projection of the point A (Fig. 4), a_0 is the correct marking; for the V projection, a_1 is used; for the P projection, a_2 is used; while a_3, a_4, etc., would be correct markings for projections on corresponding auxiliary coördinate planes that might be added. It is customary to omit the subzero on the H projections and to use a prime or accent marking instead of a subscript in the case of the V projections. These two variations are made only for purposes of speed and clarity in lettering complicated drawings, since, in the case of both the vertical and the horizontal projections, numerous additional subscripts must be used to distinguish between several projections when a single letter is used in series arrangements, or when a magnitude is projected on planes other than the coördinate planes. These conditions are not met with so much in the case of the profile and auxiliary projections because these projections are less used in obtaining solutions of problems. Every projection on a drawing must have the correct literal and numerical marking given it, or a great deal of confusion will arise.

7. Revolution of the coördinate planes

The projections of an object or magnitude can be of little practical use if the coördinate planes are kept perpendicular to each other. The three planes must be brought together into one plane, like a sheet of drawing paper, for instance. To do this, the vertical plane is assumed stationary and the two other coördinate planes are revolved into coincidence with it by using their lines of intersection with the vertical planes as axes of revolution. These two principal axes of revolution, and others to be explained later, are called ground lines. They are usually marked on the drawing in a distinctive manner, although the horizontal and vertical ones may not be so marked because of the ease with which they are recognized among other lines on the drawing. The intersection of the two principal ground lines is called the origin.

That portion of the H plane in front of the V plane is revolved down into the vertical plane and, of course, that portion of the H plane behind the V plane revolves upward into the vertical plane. In a similar manner, the profile plane in front of the vertical plane is turned toward the right while that part behind the vertical plane is turned toward the left. (See Fig. 5.) The direction of revolution is purely arbitrary but is sanctioned by long usage. In the case of the profile plane, it will often be true that the opposite direction of revolution will prove advantageous, as will be explained later. Although no inconvenience or confusion will result if the H plane is kept stationary and the V plane is revolved, in this text the V plane will be assumed stationary in all the explanations.

In the second part of the drawing of Fig. 5 there is shown the effect of the revolution of the coördinate planes on the final positions of the projections of a point. It is noted that the point A, its H projection a, and its V projection a', are all in a plane that is perpendicular to the horizontal ground line. Hence, when the H plane is revolved about the horizontal ground line as an axis, a will fall directly under a', that is, on the same perpendicular line to the horizontal ground line. This line connecting any two projections is always shown as a dotted line and is called a projecting line or, more properly, a construction line. Similarly, the points A, a', and a_2 are in a plane perpendicular to the vertical or profile ground line; hence a' and a_2 are found on the same perpendicular to this ground line after the P plane has been revolved. It will be observed, further, that only the V projection remains stationary in the revolutions of the coördinate planes.

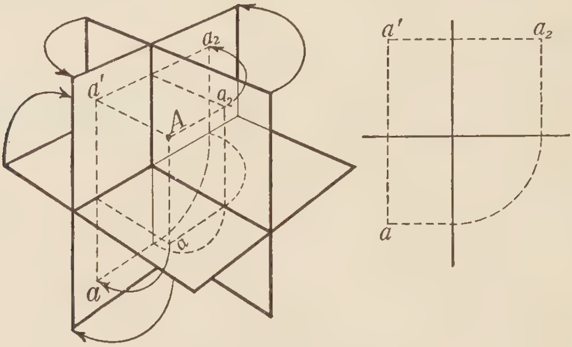


FIG. 5. Revolution of coördinate planes

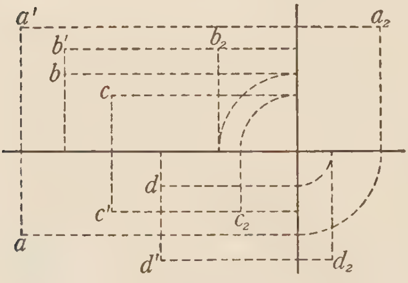


FIG. 6. Points in each of the four quadrants

The location of any projection with reference to the horizontal or vertical ground lines depends upon the quadrant in which the point or other magnitude may be placed. Fig. 6 will assist the student to fix these several relations in mind when the object is placed in each of the four quadrants.

8. Coördinates

All magnitudes are referred to the coördinate planes when their location in space is being described in words. Hence a point is said to be a certain number of linear units above or below the horizontal plane, a certain number of units in front of or behind the vertical plane, and a certain number of units to the right or left of the profile plane. This is called coördinating the point or magnitude. As in analytical and other geometries, the letters x, y, and z are used to designate the distances and directions of measurement from the point to the coördinate planes. Thus

$x =$ the distance a point is to the right or left of the P plane as x is $+$ or $-$.
$y =$ the distance a point is above or below the H plane as y is $+$ or $-$.
$z =$ the distance a point is in front of or behind the V plane as z is $+$ or $-$.

All projections of a magnitude, on the other hand, are referred to one or another of the ground lines when their positions on the coördinate planes are being described in words. An examination of Fig. 7 serves to show that definite fixed relations exist between the coördinates of a point in space and the distances and directions of its projections from the ground lines. For example, the point A, in the first quadrant, is 5 units to the left of the P plane ($x = -5$); as a result, its V and H projections are each 5 units to the left of the profile ground line. The point A is 6 units above the H plane ($y = 6$); its V and P projections are each 6 units above the horizontal ground line. The point A is 4 units in front of the V plane ($z = 4$); its H projection is 4 units below the horizontal ground line and its P projection is 4 units to the right of the profile ground line. It is customary to condense the specification of a point in space to a simple form, thus: $A = (-5, 6, 4)$, meaning that the point A has the coördinates $x = -5$, $y = 6$, and $z = 4$. The plus sign is always omitted as in other mathematical work.

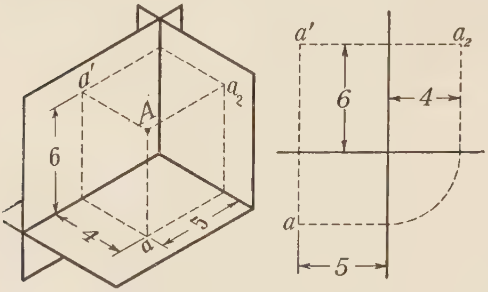


FIG. 7. Coördinate distances

A second and equally important method of indicating the position of points in space consists in referring the points to each other instead of to the coördinate planes. Thus a point is said to be to the right or left of another point, above or below it, and in front of or behind it. The relations existing between the corresponding projections of points so referred to each other are clearly shown in the example of Fig. 8,

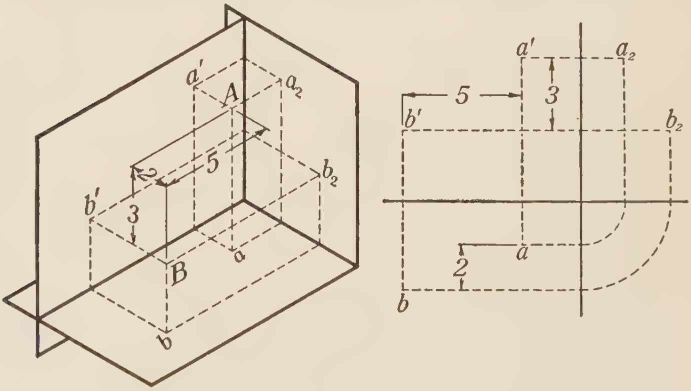


FIG. 8. Correspondence of space and projection distances

where the point A is known and the point B is described as being 5 units to the left, 3 units below, and 2 units in front of A. It should be noted that the relations of the corresponding projections are in agreement with the distances specified in space, and that these relations are not dependent upon the position of the ground lines; in fact, the ground lines need not be shown at all. All commercial or technical drawing is done upon this latter basis of relating objects and parts of objects to each other and not to the coördinate planes; hence ground lines are almost always omitted from such drawings.

9. Summary of space-projection relations of a point

In the preceding figures, points on solid objects and as entities by themselves have been shown in various positions relative to the H and V coördinate planes. Nine locations may be named as summarizing all the possible positions a point may assume in space relative to these two coördinate planes, as follows: anywhere in space in each of the four dihedral angles; in the V plane, either above or below the horizontal plane; in the H plane, either before or behind the vertical plane; and in the ground line.

These nine characteristic positions are often termed the *alphabet* of a point. They are shown in Fig. 9.

Fig. 10 shows the alphabet positions of a point in three projections instead of two, as is the case in Fig. 9. Definite fixed relations between the projections and ground lines on the one hand and the corresponding point-space positions and the coördinate planes on the other may be summarized from Figs. 9 and 10 as follows:

1. The distance from a point to the horizontal plane is always shown by the distance its V and P projections are from the horizontal ground line (Article 8).
2. The distance from a point to the vertical plane is always shown by the distance its H projection is from the hori-

zontal ground line and by the distance its P projection is from the vertical ground line (Article 8).

3. The distance from a point to the profile plane is always shown by the distance its H and V projections are from the vertical ground line (Article 8).

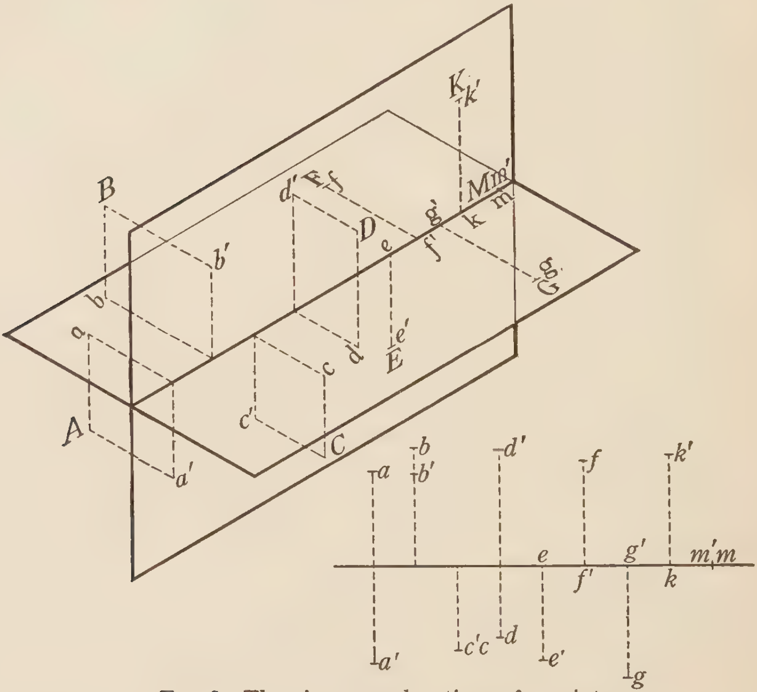


FIG. 9. The nine space locations of a point

It should be noted further from Fig. 10 that

a. The vertical and horizontal projections are always on the same vertical projecting line.

b. The vertical and profile projections are always on the same horizontal projecting line.

c. The dotted lines in the figure are mere construction devices for obtaining or maintaining these various relations and distances.

From the foregoing it is deduced that if two projections of a point are given, the point is completely determined and, conversely, if the position of a point in space with reference to the coördinate planes is known, its projections may be definitely plotted. Two projections constitute the absolute minimum necessary in any case. Solutions to formal problems are usually expressed in H and V projections, although in commercial drafting the V and P projections are quite regularly used.

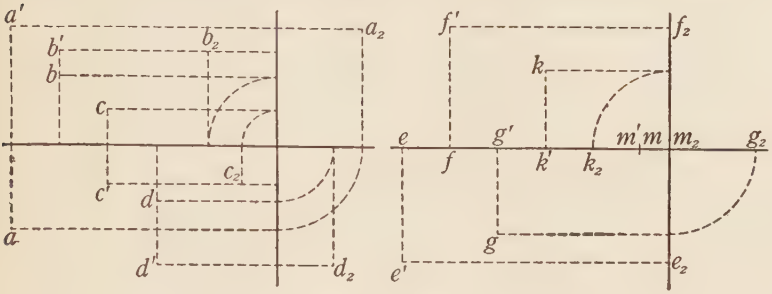


FIG. 10. Profile projections of points

10. Auxiliary coördinate planes

Often it is very desirable to show the projections of points, lines, and objects upon other planes than the three principal ones studied heretofore. The most common of these planes are those which are set up perpendicular to any one of the principal coördinate planes. They are called auxiliary coördinate planes. They may be thought of as helping planes in that they bridge the gaps often encountered in drawing when only the regular planes are used. Fig. 11 shows one such plane perpendicular to the horizontal plane. It is called *auxiliary coördinate plane number 3* or simply *coördinate plane 3* (see Article 3).

The intersection of this auxiliary coördinate plane 3 with the H plane constitutes a third ground line about which the auxiliary plane is revolved, in either direction, into the horizontal plane. This new ground line is marked *0–3* in the manner shown. The numeral *0* indicates that the auxiliary plane is perpendicular to the zero, or H, plane. The numeral *3* indicates that the auxiliary plane is numbered 3.

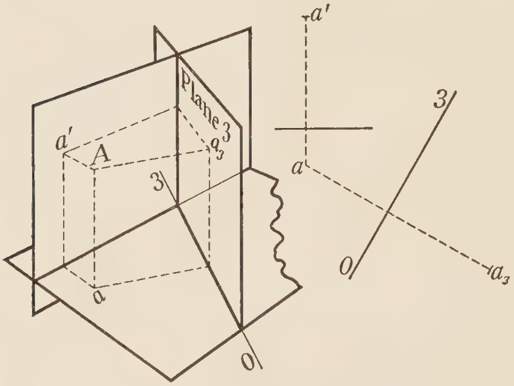


FIG. 11. Auxiliary coördinate plane 3

A point A is projected upon plane 3 by means of a projector through the point perpendicular to the plane in the same way that it would be projected upon the H, V, or P plane, and the projection is marked a_3 (see Fig. 11). When plane 3 has been revolved into the horizontal plane, which in turn revolves into the V plane, there is obtained the view shown in the right half of the figure. Attention is directed to the fact that a and a_3 are on the same perpendicular to the ground line *0–3* for the identical reasons that a and a' are on the same perpendicular to the horizontal ground line (see Article 7). The distance from a_3 to *0–3* must be equal to the distance from a' to the horizontal ground line, each distance representing the same thing, namely, the distance the point is located above the H plane.

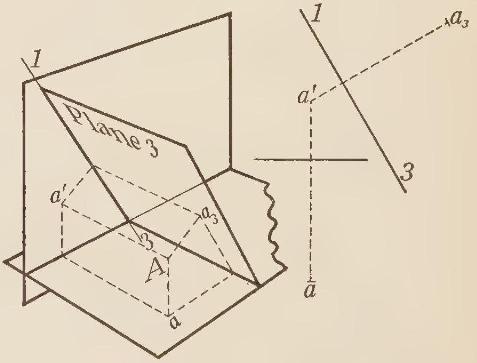


FIG. 12. Auxiliary coördinate plane 3

Fig. 12 shows the conditions when the auxiliary coördinate plane is set up perpendicular to the V plane. In this case the ground line must be marked *1–3* to indicate that plane 3 is perpendicular to the V plane. Since the V plane is stationary, only one revolution is involved in bringing the auxiliary plane into coincidence with the vertical plane. Here the distance from a_3 to *1–3* is equal to the distance a is from the horizontal ground line, each being the distance the point A is from the vertical plane.

Auxiliary planes may also be set up perpendicular to the profile plane, in which case the ground lines are marked *2–3*. Here, as in the first instance cited, two revolutions are involved in bringing the auxiliary plane into coincidence with the vertical plane (see Fig. 13). The distance to be laid off from the ground line *2–3*, in this case, is the distance the H or V projection is to the left or right of the vertical ground line, that is, the distance from the point A to the profile plane.

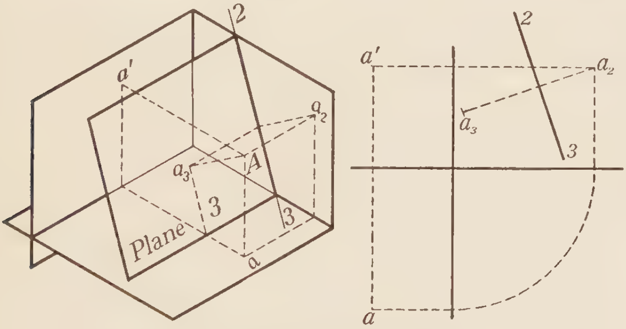


FIG. 13. Auxiliary coördinate plane 3

In each of the three cases cited above and in all instances where an auxiliary coördinate plane is used, the projecting line, upon which the measurement is laid off, must always be perpendicular to the new ground line, and must pass through the projection of the point on the coördinate plane to which the auxiliary plane is set up perpendicular. Auxiliary planes may be revolved in either of the two possible directions.

From the foregoing and from a careful examination of Figs. 11, 12, and 13, the following statements of fact are derived:

1. An auxiliary coördinate plane and the principal plane to which it is perpendicular form a new set of coördinate planes having the same properties and relationships to each other as do any two of the principal coördinate planes.

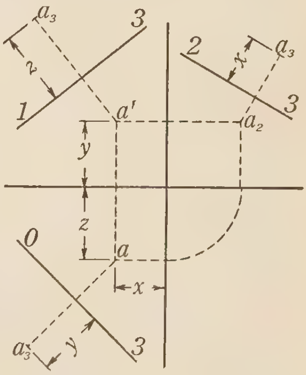


FIG. 14. Coördinate distances

2. The projection of a given point on an auxiliary coördinate plane is found on a line perpendicular to the new ground line, through the projection of the point on the principal plane to which the auxiliary plane is set perpendicular.
3. The distance from the auxiliary projection of a given point to its reference ground line is the same as the distance another projection of the point is from its reference ground line, when the two projections compared are on planes perpendicular to the same coördinate plane. This important fact is illustrated by the projections shown in Fig. 14.

CHAPTER II

PROJECTIONS OF LINES

11. Straight lines

A line is the path of a moving point. A straight line is the path of a point that moves without change of direction. The direction of a straight line is completely determined when two points on it are definitely located. It is only necessary, therefore, to establish the H, V, and P projections of any two chosen points on a given line, such as A and B in Fig. 15, for instance, and then to join the corresponding projections of the points by straight lines (projections) in order to represent the given line completely. Of course, AB is only a segment of an infinite line in space — all straight lines being infinite in extent — but it is customary to speak of AB as a definite and determinate line, to be produced at will by the simple expedient of extending its several projections simultaneously in corresponding directions.

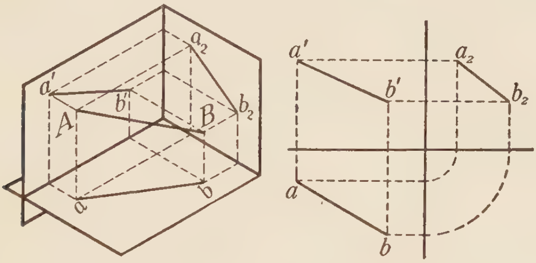


FIG. 15. Projections of a straight line

12. Curved lines

A curved line is the path of a point that moves in a continually changing direction. Curved lines may not be represented so easily as straight lines. A number of points sufficiently close together to permit of smooth curved lines being drawn through their respective projections must be projected upon the coördinate planes, so that any intermediate points may be located with a fair degree of accuracy (see Fig. 16). The curved line shown differs from the straight line of Fig. 15 in that it may not be extended at will. A curved line should be designated and read by means of two or more letters, thus: "line AB, with projections *ab*, *a′b′*, and a_2b_2," or "line ABCD," with corresponding projections.

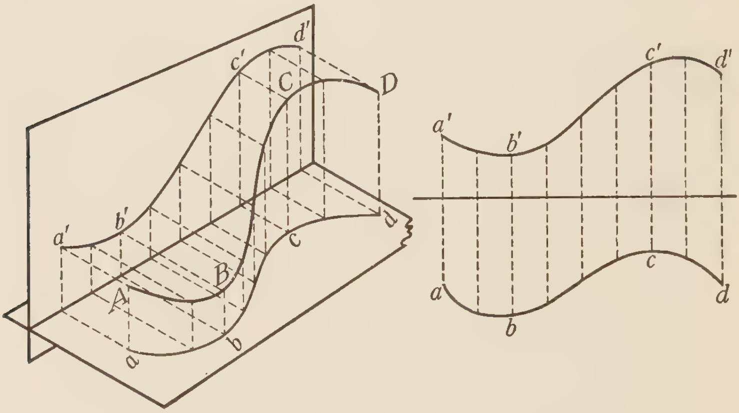


FIG. 16. Projections of a curved line

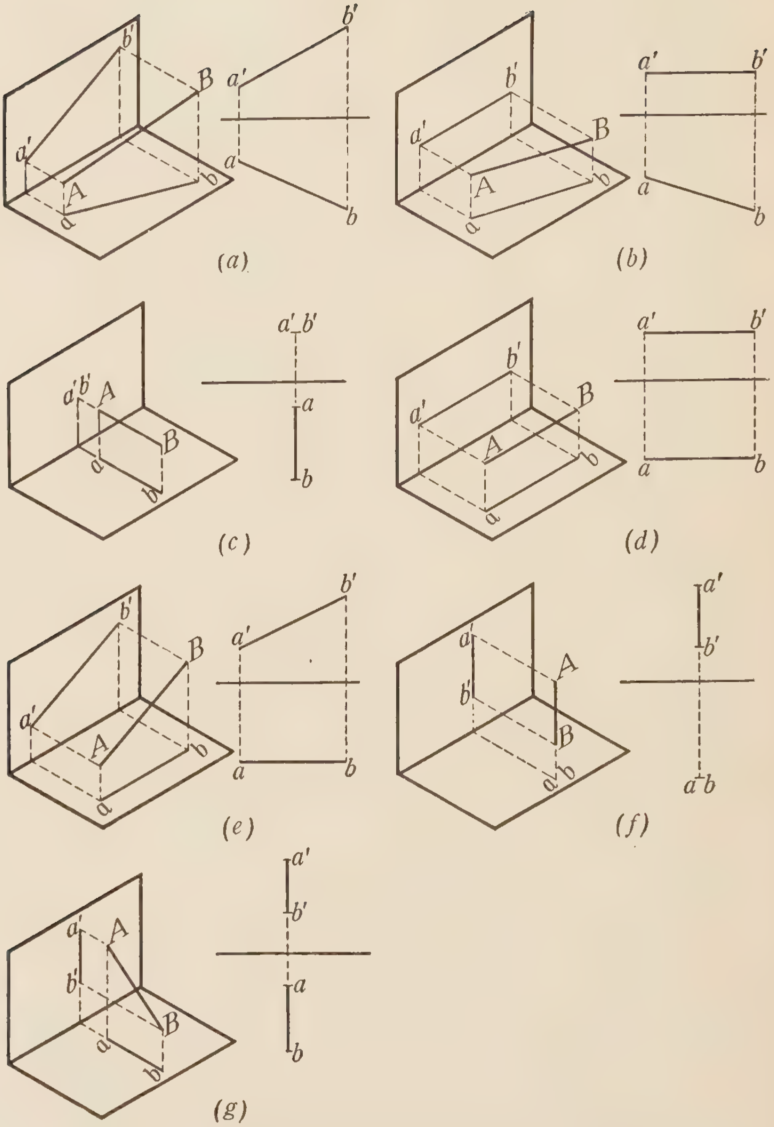
13. Lines versus projections

Difficulty is always experienced by the beginner in distinguishing properly between a given line in space and its projections on paper, especially in his oral references to them. This trouble is overcome entirely by strict adherence to the simple rule that the representations of a space line on the coördinate planes must be referred to as projections instead of as lines. Thus it is correct to say "the H projection *ab*," whereas the simple phrase "the line *ab*," referring to the horizontal projection, is bound to be ambiguous, because anyone hearing the expression is in doubt whether the line AB in space is meant or its horizontal projection. Not so much difficulty is experienced in distinguishing between a line in space and its V, P, and auxiliary projections, on account of the accents and subscripts used with these projections. It must be understood, also, that the location, inclination, and other space characteristics of lines or any other magnitudes are always described by reference to the coördinate planes, and not to the ground lines, as is the case when their projections are being discussed.

14. Relations of lines to the coördinate planes

A line of definite length may be placed in seven characteristic or typical positions relative to the three principal coördinate planes. One or more of three descriptive words — "inclined," "parallel," or "perpendicular" — must be used to indicate the relations of each of these lines to the coördinate planes. Six of the seven typical lines are parallel to one or another of the three principal coördinate planes. This fact is of much importance in construction work and is made the basis of the following summary, by first bringing the general inclined line parallel to the H plane and turning it through an angle of 360°, then parallel to the V plane and turning it 360°, and finally parallel to the P plane, and listing in order the characteristic positions of the line as they occur in the revolutions. A line in space may be

1. Inclined to all three coördinate planes; its H and V projections will be inclined to the horizontal ground line (Fig. 17 *a*).
2. Parallel to the H plane and inclined to the V and P planes; its V projection will be parallel to, and its H projection will be inclined to, the horizontal ground line (Fig. 17 *b*).
3. Parallel to the H and P planes and perpendicular to the V plane; its V projection will be a point, and its H projection will be perpendicular to the horizontal ground line (Fig. 17 *c*).



(a) (b) (c) (d) (e) (f) (g)

FIG. 17. The seven space directions of a line

4. Parallel to the H and V planes and perpendicular to the P plane; its H and V projections will be parallel to the horizontal ground line (Fig. 17 *d*).
5. Parallel to the V plane and inclined to the H and P planes; its H projection will be parallel to, and its V projection will be inclined to, the horizontal ground line (Fig. 17 *e*).
6. Parallel to the V and P planes and perpendicular to the H plane; its H projection will be a point, and its V projection will be perpendicular to the horizontal ground line (Fig. 17 *f*).
7. Parallel to the P plane and inclined to the H and V planes; its H and V projections will be perpendicular to the horizontal ground line (Fig. 17 *g*).

Every student should determine for himself the profile projection of each of these characteristic lines. This may be done by finding the profile projections of the ends of each line, as explained in Article 7, and then joining them by a straight line. The procedure is shown for each characteristic line in Fig. 18.

These seven typical positions of a line taken together constitute what is often termed the *alphabet* of a line. The term "alphabet" may be restricted to mean that the included lines are limited in length and position to one definitely stated quadrant, as is done in the case of the seven drawings of Figs. 17 and 18, or it may be used in the more general sense to include these typical lines without reference to their lengths or the quadrants in which they may be found. The latter usage is preferred. The location of the profile plane has no effect upon the alphabet characteristics of a line in either case.

15. H-, V-, and P-parallels

It will be noted from the foregoing summary of the relations of lines to the coördinate planes that numbers 2, 3, and 4 specify lines parallel to the H plane. Any line so related to the H plane is called an H-parallel. Similarly, numbers

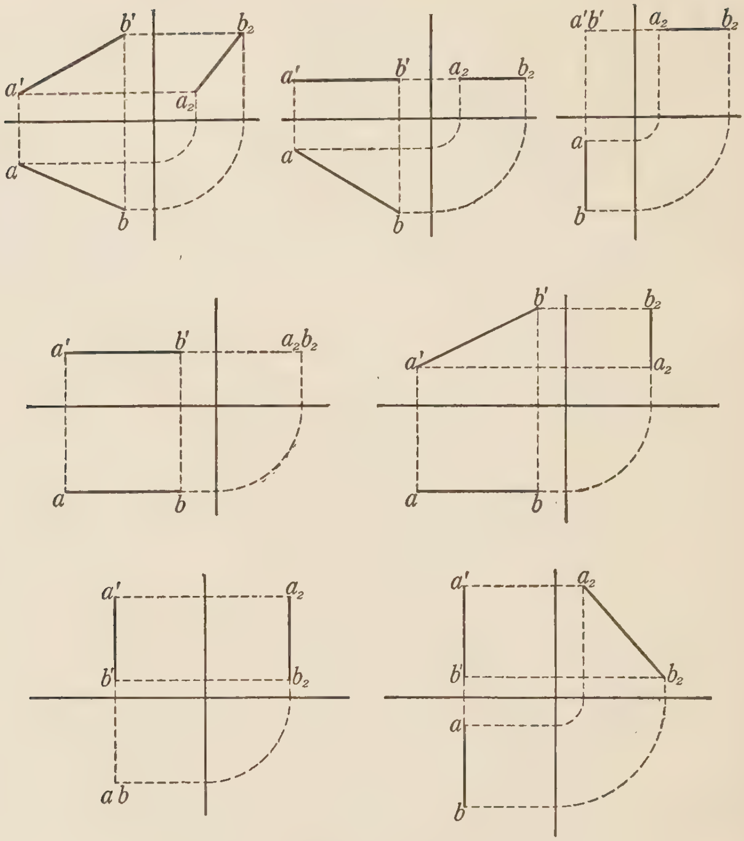


FIG. 18. Profile projections of lines

4, 5, and 6 specify lines parallel to the V plane; they are called V-parallels. Numbers 3, 6, and 7 specify lines parallel to the P plane; they are called P-parallels. Lines specified in numbers 3, 4, and 6 fall under each of two designations and may be spoken of in either of two ways.

16. Intersecting lines

Two intersecting lines contain a common point in space. This point must have its H projection on the H projection of each line — therefore at their intersection. Similarly, the

V projection of this point must be at the intersection of the V projections of the lines. These two points of projection must be on the same perpendicular to the ground line (Article 7). Hence, to draw a line CD intersecting a line AB at a point C, it is only necessary to assume the projections of the point C on the respective projections of AB and in the same perpendicular to the ground line, and then to draw the projections of CD in any desired direction, as shown in the upper drawings of Fig. 19. The point A or the point B might have been taken as the point of intersection and the line CD drawn through it in any desired direction, as shown in the lower half of the figure. The profile projection of the line AB might have been used equally well with either of the other two projections to determine the point of intersection.

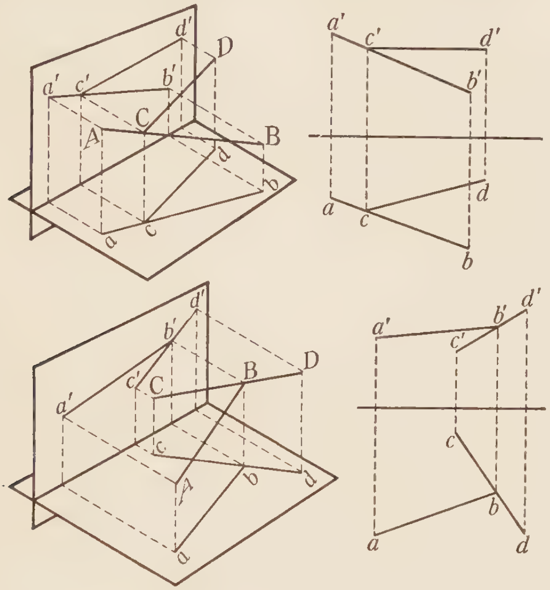


FIG. 19. Intersecting lines

Conversely, to ascertain if two lines already drawn do actually intersect, it is only necessary to observe whether the point of crossing of the H projections, produced if need be, is on the same perpendicular to the ground line as the point of crossing of the V projections.

If, in the foregoing cases, one of the lines had been a P-parallel, then the profile projections of the lines would have been utilized to determine whether or not the lines actually intersect and are correctly drawn.

17. Projections of lines on auxiliary coördinate planes

In Article 10 it was shown how a point already projected upon the principal planes may also be projected upon an auxiliary plane perpendicular to one of the coördinate planes, by laying off certain distances on projecting lines drawn through the proper projections of the points and perpendicular to the new ground line. A line may be projected upon such a plane by projecting its two end points in the manner referred to and joining the resulting projections with a straight line. In Fig. 20 the line AB has been projected upon an auxiliary coördinate plane 3, which is set up perpendicular to the H plane. The auxiliary projection is marked a_3b_3. Plane 3 may be revolved in either direction about *0–3* as an axis into the horizontal plane. The right half of the figure shows the relative locations of the projec-

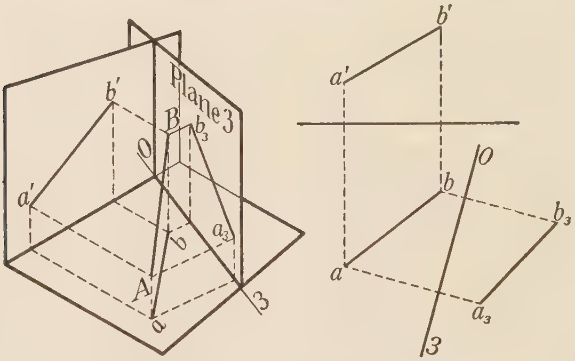


FIG. 20. Plane-3 projection of a line

tions after the revolution of each of the coördinate planes has taken place. It should be pointed out again that a' and a_3 are at equal distances from their respective ground lines. This is true because these lengths both represent the distance of the point A from the H plane, as the pictorial drawing of the figure shows. Likewise, b' and b_3 are at the same distances from their respective ground lines.

Fig. 21 shows the line AB projected upon an auxiliary coördinate plane 3 which is perpendicular to the V plane.

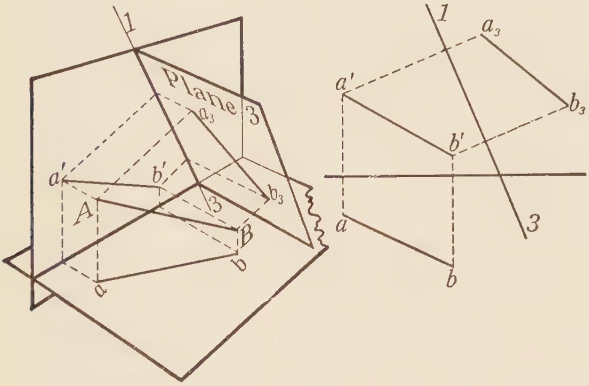


FIG. 21. Plane-3 projection of a line

Here a and a_3 are at equal distances from their respective ground lines because these lengths represent the distance of the point A from the V plane. The corresponding projections of B have a similar relation to each other.

An auxiliary plane 3 may be set up perpendicular to the profile plane and the sub–3 projections obtained in a manner similar to that of the preceding examples.

In each case cited above and in all other cases where a plane 3 is used, the auxiliary projections are always found on perpendiculars to the new ground lines and through the projections on that coördinate plane to which the auxiliary plane has been set perpendicular. Later it will be explained how a second auxiliary plane may be set up perpendicular to any one of the three auxiliary planes previously mentioned. It will be found that the rule just stated will hold with respect to the projections on this new auxiliary plane and its ground line.

18. Piercing points of lines

It has been stated that a line is infinite in length, although by common usage a segment of it is termed the line. Hence a line inclined to all three principal planes may be extended into as many as three quadrants or it may be confined to only one. When a line is extended indefinitely, it will pierce at least one and sometimes two or three of the principal coördinate planes at points called the *piercing points* of the line. These points are of such importance that the letters H, V, and P are used exclusively to indicate their location on the respective coördinate planes. The projections of these three points for the general inclined line AB of Fig. 22 are determined as follows:

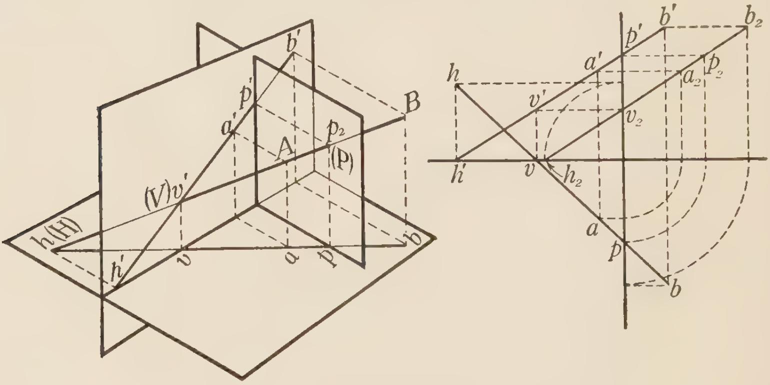


FIG. 22. Piercing points of a line

H piercing point. The V projection of the H piercing point must lie on the horizontal ground line, because the point itself is on the H plane; it must also lie on the V projection of the line, hence at their intersection, *h'*.

The H projection of the H piercing point must lie on the H projection of the line; it must also lie on the perpendicular to the horizontal ground line through h', hence at their intersection, h. The profile projection, h_2, of the H piercing point is found by means of h and h' or by extending the profile projection of AB to the horizontal ground line, as shown in the figure.

V piercing point. The V piercing point of a line may be found by methods of locus reasoning similar to that used for the H piercing point. The sequential steps are summarized in rule form thus: Extend the H projection of the line until it intersects the horizontal ground line to obtain v. Erect a perpendicular to the ground line at v until it intersects the V projection of AB at v'.

The profile projection, v_2, is obtained from v' and v or by extending a_2b_2 to the vertical ground line, as shown in the figure.

P piercing point. The horizontal and vertical projections of all points in the P plane must be in the vertical ground line. The vertical projection of the P piercing point must also be on $a'b'$. Therefore p' must be at the intersection of $a'b'$ and the vertical ground line. Likewise, the horizontal projection, p, of the P piercing point must be at the intersection of ab with the vertical ground line. The profile projection, p_2, of the P piercing point may be obtained by use of p' and p or by means of either of these projections and a_2b_2, as shown in the figure.

In finding the H and V piercing points of a line that is parallel to the profile plane and inclined to the other coördinate planes, difficulty is experienced in applying the foregoing rules because the H and V projections of the line are perpendicular to the ground line. This difficulty is overcome by first obtaining the profile projections, h_2 and v_2, of the H and V piercing points of the line. In Fig. 23 the line AB has been projected upon the P plane as a_2b_2, and h_2 and v_2

have been obtained as shown. The vertical projection of the point V is on $a'b'$ produced and also on a horizontal with v_2, therefore at v'. The horizontal projection of the point H is on ab produced and also at a distance from the horizontal ground line equal to the distance of h_2 from the vertical ground line, therefore at h. The projections h' and v are on the horizontal ground line as in Fig. 22.

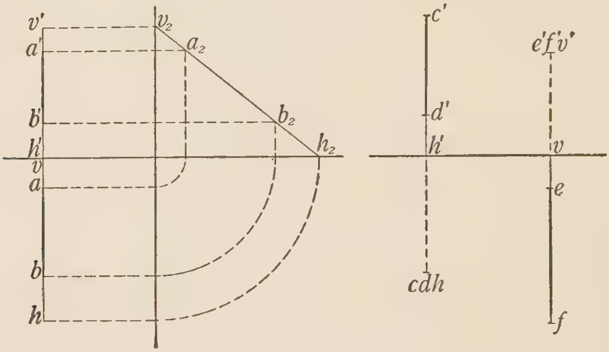


FIG. 23. Piercing points of special lines

Fig. 23 also shows the method of obtaining the piercing points of lines that are perpendicular to the H and V planes, respectively, as indicated for the lines CD and EF in the second part of the figure.

19. True length of a line

It is readily understood that the orthographic projection of any line on a coördinate plane to which the line is parallel shows the true length of that line. Of the seven lines which may be drawn in characteristic positions relative to the H, V, and P planes, six are coördinate-plane-parallels. Their true lengths are shown on the principal coördinate planes to which they are parallel (see Figs. 17 *b* to 17 *g*, inclusive). Only one line of the seven is inclined to all the coördinate planes. In Fig. 24 an auxiliary coördinate plane 3 has been set up parallel to this inclined line. This is done by making

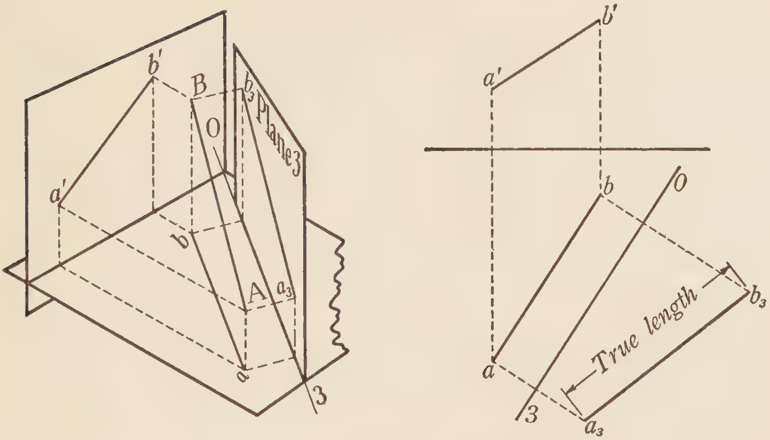


FIG. 24. True length of an inclined line

0–3 parallel to *ab*. The line is now a plane-parallel with reference to plane 3. The projection a_3b_3 of the line AB on plane 3, therefore, is the true length of the line AB. This use of auxiliary coördinate planes constitutes the first reason for their existence.

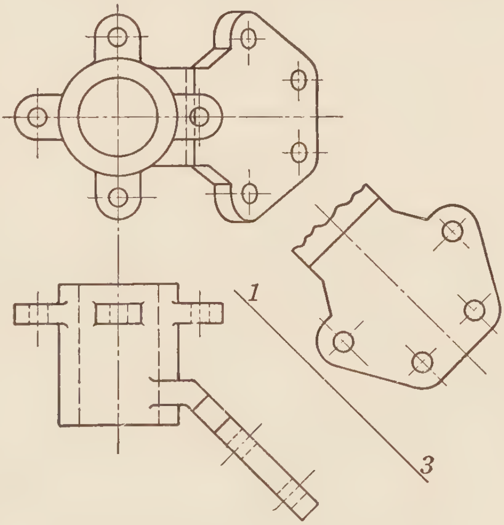


FIG. 25. Auxiliary view of bracket

A simple extension of the principle just explained to a common practical use is shown in Fig. 25, where an auxiliary coördinate plane 3 has been set up, not parallel to one line only on the object, but parallel to several lines, both straight and curved. The projections of these lines upon plane 3, as shown in the auxiliary view in the right center portion of the drawing, represent not only the true lengths of the lines but also their correct angular and other space relations to one another, as well as the true shape of the face of the object which is projected. The ground line *1–3* is usually omitted in commercial drafting.

20. Angle between intersecting lines

If two intersecting lines are each parallel to the same principal coördinate plane, the true angle X between them is shown by the angle their projections make with each other on that coördinate plane (see Fig. 26). Their projections on either of the other two principal coördinate planes fall in a single straight line parallel to a ground line, as the figure illustrates for the vertical projections.

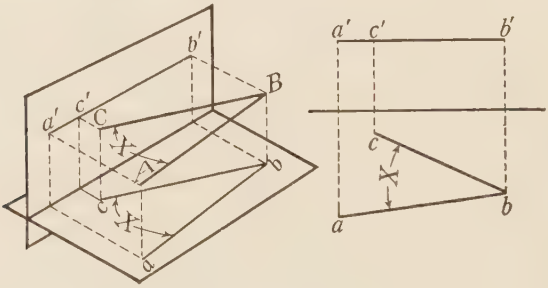


FIG. 26. Angle between two lines

Of course, either one or both of two intersecting lines may not be parallel to a principal coördinate plane, in which case the use of the auxiliary coördinate planes will prove effective. The arrangement of the auxiliary planes, however, may be rather involved; hence discussion of these conditions will be deferred to later paragraphs. A simple example is shown in Fig. 27 to emphasize the importance of using auxiliary coördinate planes when none of the projections on the H, V, or P planes give the true value of the angle X. Because the projections of the two lines on one coördinate plane happen to fall in a straight line, in this case *acb*, it is possible to set up the auxiliary plane directly.

A single exception to the general rule that both lines must be parallel to a coördinate plane in order that the true angle between them may show on the plane, occurs in the case of two lines in space which make an angle of 90° with each other. If only one of two such lines is parallel to a coördinate plane, the true angle of 90° shows on that plane. Fig. 28 illustrates the reason for this exception to the general rule.

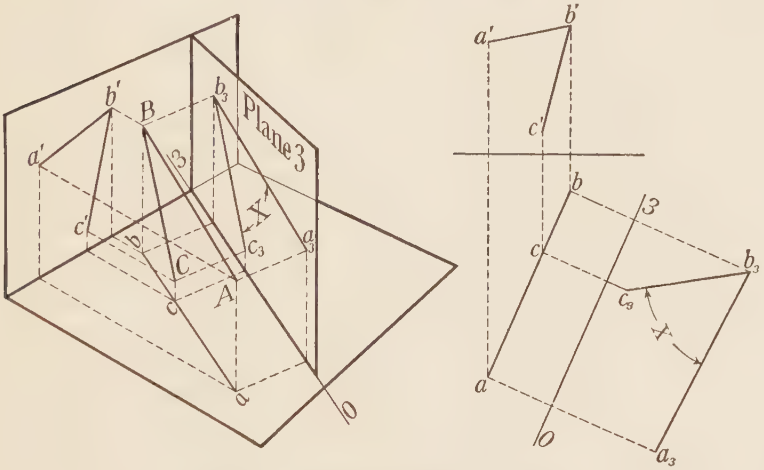


FIG. 27. Angle between two lines

In the figure the face ABCD of the solid object is drawn parallel to the vertical plane; hence each line in this face of the object is parallel to the V plane. The face CBGK is perpendicular to the vertical plane and projects upon it in a straight line, *b′c′* or *g′k′*; hence any line in this face of the object will project upon the vertical plane in *b′c′*.

First consider the perpendicular lines AB and BC, both of which are parallel to the V plane. Their vertical projections, $a'b'$ and $b'c'$, make an angle of 90° with each other; their horizontal projections, ab and bc, coincide. These conditions are in accord with those of the general rule expressed in the first paragraph of this article. Next consider the perpendicular lines AB and BE, only one of which, AB, is parallel to either coördinate plane shown in the figure. The vertical projections, $a'b'$ and $b'e'$, of the two lines show the true angle of 90° between the lines, while their horizontal projections make an unknown angle with each other. Similarly, the perpendicular lines AB and BF have their vertical projections at 90° to each other and their horizontal projections at an unknown angle. Therefore, it is repeated, when one of two intersecting 90° lines is parallel to a coördinate plane, the projections of the two lines on that coördinate plane will show the true value of the 90° angle. This important relation permits us to draw a great many of the projections of solid objects directly without the use of auxiliary planes or other devices. Probably no other fact of descriptive geometry has more manipulative and practical value to the student. He should master its every detail.

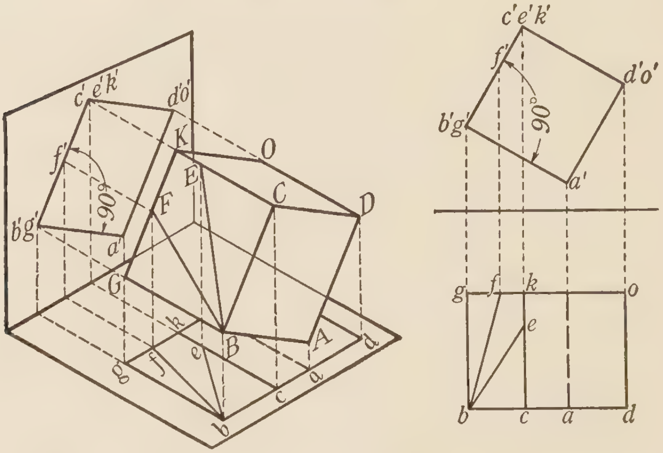


FIG. 28. Projections of 90° angles

21. Parallel lines

It may be stated as a proposition requiring proof that the corresponding orthographic projections of parallel lines are always parallel; and, conversely, if all the corresponding projections of lines are parallel, the lines themselves are parallel. The importance of these relations in descriptive-geometry work cannot be overstated.

Fig. 29 shows two parallel lines, AB and CD, in the first quadrant. Through each line has been passed a plane perpendicular to the H plane. These planes are parallel for the reason that each contains two lines that are parallel, respectively, to two lines of the other — AB is parallel to CD and A*a* is parallel to C*c*. The intersections of these two planes with the H plane must be parallel, by a well-known principle of solid geometry. These intersections are in reality the H projections of the lines AB and CD, which projections, therefore, are parallel. The same proof is applicable to any other set of projections of the two lines.

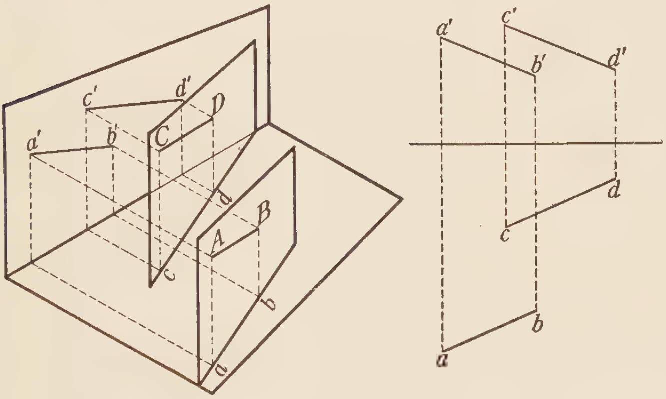


FIG. 29. Projections of parallel lines

Care should be taken in applying the converse of this principle not to invalidate its use by restricting the projections of the lines to two coördinate planes. An exception will seem to occur in the case of lines parallel to the P plane, where it is necessary to obtain the P projection of the lines before the H and V projections may be drawn.

Perhaps the most common application of the principle explained in the preceding paragraphs is in connection with the drawing of a line in some indicated direction through a given point. The direction is best expressed in terms of parallelism to some line already drawn or to be determined. If, for example, in the first drawing of Fig. 30, it is known that the two intersecting lines AB and BC are the two sides of a rectangle, the other two sides may be drawn by making their projections parallel to the corresponding projections of AB and BC respectively.

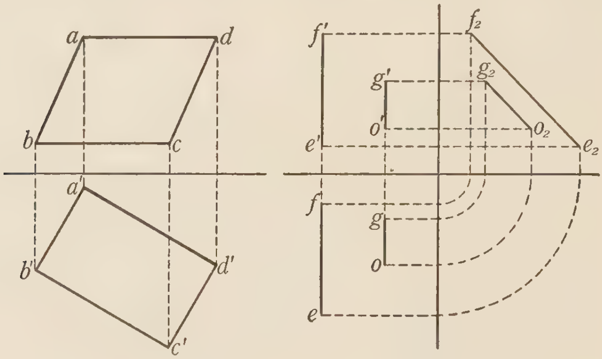


FIG. 30. Projections of parallel lines

When a given line is parallel to the profile plane and inclined to the other two coördinate planes, the profile projection of the line to be drawn parallel to the given line must be obtained before its H and V projections can be found. In the second part of Fig. 30 is shown the procedure for drawing a line parallel to EF through the point O. The line EF and the point O are shown on the profile plane as e_2f_2 and o_2 respectively. Through o_2 is drawn o_2g_2 parallel to e_2f_2, and of any length. The H projection of the point G is on a line drawn through o parallel to ef. Its exact location is fixed by the usual return* constructions between the profile and horizontal projections of a point. Similarly, the V projection of the point G is on a line through o' parallel to $e'f'$. It is determined definitely by the usual horizontal return construction from the profile projection, in this case from g_2. The line OG is now known to be parallel to the line EF.

22. Perpendicular lines

For purposes of definiteness only, the conception of perpendicular lines will be limited in this text to intersecting lines. A line lying in a plane that is perpendicular to another line will not be treated as perpendicular to the second line unless the two lines actually intersect, although, in a purely mathematical sense, they are so considered. Perpendicular lines do not have their corresponding projections perpendicular, except as explained in Article 20.

23. Line representation of plane and solid figures

In plane geometry many figures were dealt with, such, for instance, as the triangle, square, rectangle, circle, and ellipse. These figures were always drawn on what we now know as the vertical or horizontal plane; that is, the figures were assumed in or parallel to one of these planes. It is necessary in orthographic projection to represent these plane figures by several views and in whatever position in space they may be found. They will not, of course, project on any coördinate plane in true size or shape unless they are parallel to the plane. This change of shape and size in projecting plane figures is called foreshortening, because, if there is any change at all, a given line or area will always appear less rather than greater in extent when projected. The first drawing of Fig. 31 illustrates this point in the case of a parallelogram.

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* The process of locating the H and V projections of a point from and after either the profile or auxiliary projections have been determined is called "returning the point."

All plane geometrical figures may be represented upon the coördinate planes by treating them as collections of straight or curved intersecting lines. However, without further derivation of principles and construction methods the student is limited in his solutions of problems to plane geometrical figures that are parallel to some coördinate plane. The plane surface represented in Fig. 31 and referred to in the preceding paragraph could not have been projected from its oblique position in space without the aid of principles and methods developed in Chapter IV.

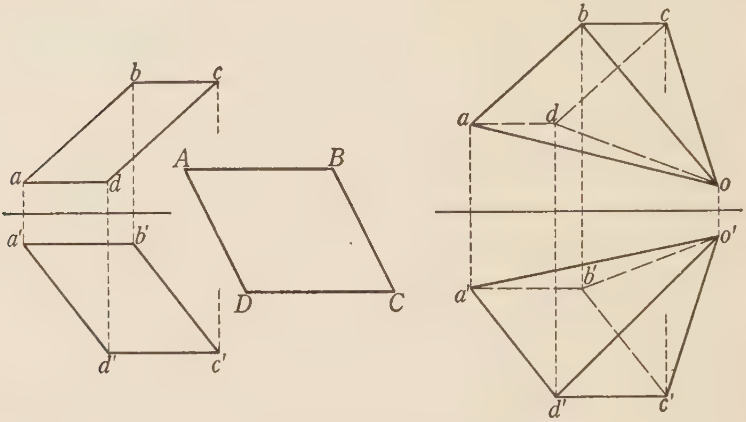

FIG. 31. Foreshortening; invisible lines

Solids are bounded by collections of plane geometrical figures, by curved surfaces, or by combinations of the two. The second drawing of Fig. 31 represents a pyramid bounded by four plane triangles and one parallelogram. The two views of the solid were obtained by projecting each edge line upon the coördinate planes either as a separate entity or in conjunction with other edge lines of one or another of the bounding plane figures. Explanation of the method of constructing the projections must be deferred to Chapter IV. The item of importance to be noted here is that the representations of both plane and solid geometrical magnitudes are made by means of the projections of lines that are either edge or contour lines on the object, as the case may be.

The bounding surfaces of objects are considered opaque. This means that in some views certain lines or portions of them may be hidden from the observer by intervening opaque surfaces. These lines are called hidden or invisible lines. Such lines are indicated on the drawing by means of a broken-line or dash-line convention. This has been done in the case of the pyramid of Fig. 31. It should be noted that the same line may not appear invisible in both views. Only practice and careful analysis of the magnitudes involved and of their location in space can make the draftsman expert in determining the invisible lines on an object as seen from different points of view. The importance of the invisible lines in reading and interpreting a drawing and in determining, from an unlettered drawing, the quadrant in which an object has been placed, cannot be too strongly emphasized.

Since the determination of invisible lines on an object depends upon the location of the point of sight for each view, it is arbitrarily assumed that the observer is located above the object and the H plane when horizontal projections or views are being drawn, in front of the object and the V plane when vertical projections or views are being drawn, and to the left of the object and the P plane when profile views are projected. This rule is not always adhered to in the case of the profile plane, as will be explained later.

CHAPTER III

REPRESENTATIONS OF PLANES

24. Methods of determining and representing planes

A plane is generated by moving a straight line so that it constantly touches two other parallel or intersecting straight lines. A plane is completely determined by a point and a line, by two intersecting or parallel lines, or by three points.

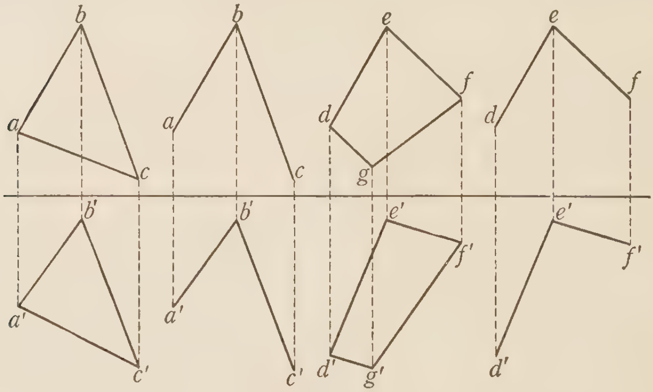

FIG. 32. Plane of surface determined by two lines in it

A plane is usually represented on an orthographic drawing by two or more intersecting lines or by two parallel lines. In Fig. 32 the triangle ABC and the quadrilateral DEFG are definite plane surfaces and are completely represented by the two projections shown. Any two lines, such as AB and BC of the triangle or DE and EF of the quadrilateral, would be sufficient to fix definitely the space characteristics of the planes, but any third line joining the two chosen lines may be substituted for either of them.

Oftentimes it is advantageous to represent planes by two specially chosen intersecting or parallel lines called *traces,* that is, by the lines in which the planes, when produced, intersect the coördinate planes. In this text both the method of representation by traces and that by lines are employed.

25. Traces of planes

If any oblique plane in space is extended indefinitely, it will intersect all the principal and auxiliary coördinate planes. These lines of intersection are called the H, V, P, and auxiliary traces, depending upon the coördinate plane on which any particular trace is drawn (see Fig. 33).

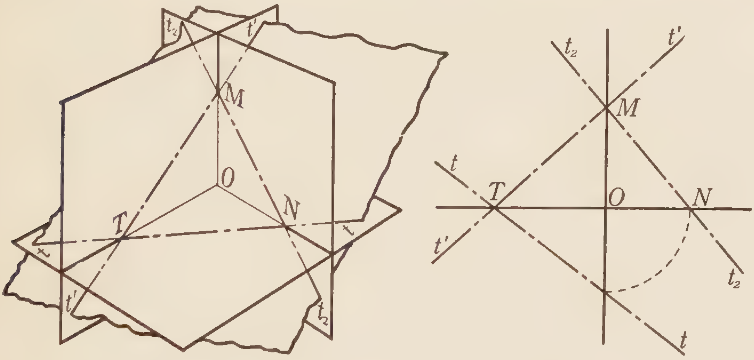

Fig. 33. Trace representation of a plane

Traces are distinguished from projections of lines and solids by means of a conventional dash-dot line and by distinctive letters which are put on each end of this conventional line. A small letter with a subscript numeral corresponding to the number of the coördinate plane on which the trace appears is put on the end of each trace of the plane, except the vertical trace, where the prime marking is substituted for the subscript. In addition, a capital letter is put at the intersection of the horizontal and vertical traces when these traces are produced to the horizontal ground line. As a rule, the last letters of the alphabet are used for designating planes. Since planes are infinite in extent, their traces may be extended at will. Traces are lettered as shown in the drawings of Figs. 34 and 35. Traces are read by means of the two small letters placed at their ends, except when they extend across the horizontal ground line; then the capital and one small letter are used for designating the H and V traces. The capital letter is always read first.

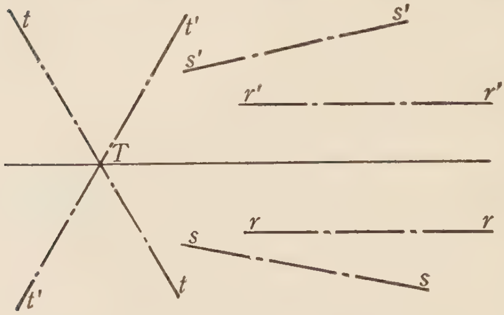

FIG. 34. Lettering traces of planes

26. Intersection of traces on ground lines

Further reference to the first part of Fig. 33 discloses the fact that the three traces of the plane there represented intersect in pairs on one or another of the ground lines. This must be so in every case where the plane itself intersects the ground lines. It should be observed that a trace of a plane is the locus of all points common to the plane and the corresponding coördinate plane. For example, *t't'* contains all the points that are in the plane T and that are at the same time in the V plane.

It should be noted, further, from Fig. 33, that in the revolution of the H and P planes no change takes place in the location of the point of intersection of the H and V traces at *T*, nor does any change occur in the position of the point of intersection of the V and P traces on the vertical ground line. However, the point of intersection of the H and P traces is moved. It revolves to the right with the profile plane to determine a point on the horizontal ground line through which the profile trace must pass, and it revolves down into the vertical ground line to determine a point through which the H trace must pass. The distance of each of these new positions of the point from the origin after the planes have been revolved is the same as the distance of the point of intersection from the origin before the planes were revolved. This relation is shown by the 90° construction arc joining the H and P traces. Fig. 35 shows the relation of the three principal traces and their points of intersection on the ground lines when an oblique plane is placed in the different quadrants and at various inclinations to the coördinate planes. Discussion of the traces of general planes on the auxiliary coördinate planes will be found in Article 30.

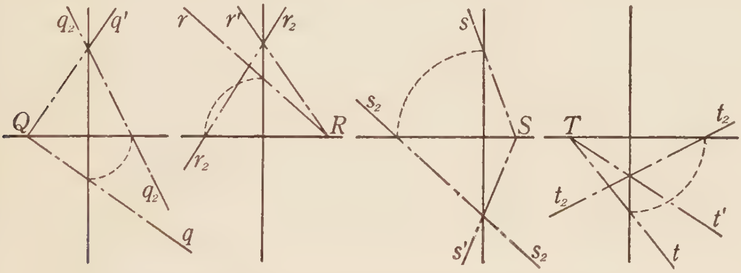

FIG. 35. Traces of planes in the four quadrants

27. Relations of general planes to the coördinate planes

A plane of indefinite extent may be placed in seven characteristic positions relative to the three principal coördinate planes. One or more of three descriptive words—"inclined," "parallel," and "perpendicular"—must be used to indicate each of these positions. A plane may be

1. Inclined to all three coördinate planes; its H and V traces will be inclined to the horizontal ground line (Fig. 36 *a*).
2. Perpendicular to the H plane and inclined to the V and P planes; the V trace will be perpendicular to, and the H trace will be inclined to, the horizontal ground line (Fig. 36 *b*).

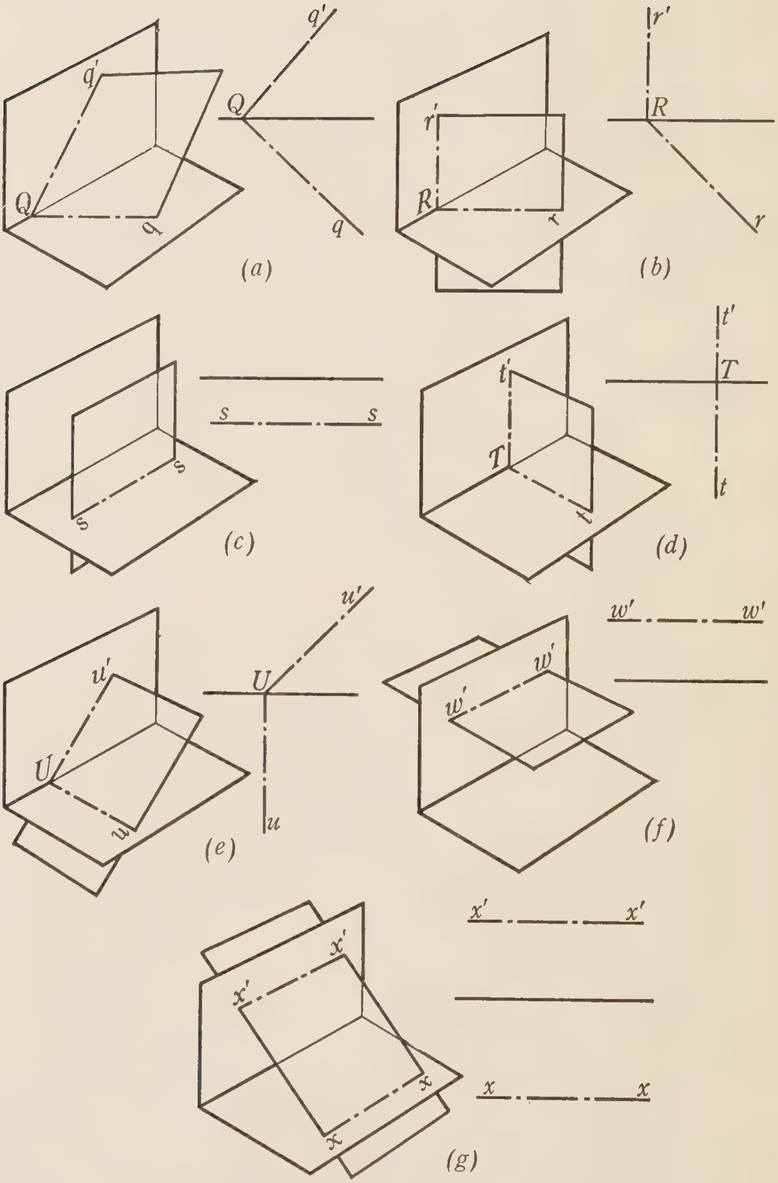

FIG. 36. The seven space positions of a plane

3. Perpendicular to the H and P planes and parallel to the V plane; the H trace will be parallel to the horizontal ground line, and the V trace will be at infinity (Fig. 36 *c*).
4. Perpendicular to the H and V planes and parallel to the P plane; the H and V traces will be perpendicular to the horizontal ground line (Fig. 36 *d*).
5. Perpendicular to the V plane and inclined to the H and P planes; the H trace will be perpendicular to, and the V trace will be inclined to, the horizontal ground line (Fig. 36 *e*).
6. Perpendicular to the V and P planes and parallel to the H plane; the V trace will be parallel to the horizontal ground line, and the H trace will be at infinity (Fig. 36 *f*).
7. Perpendicular to the P plane and inclined to the H and V planes; the H and V traces will be parallel to the horizontal ground line (Fig. 36 *g*).

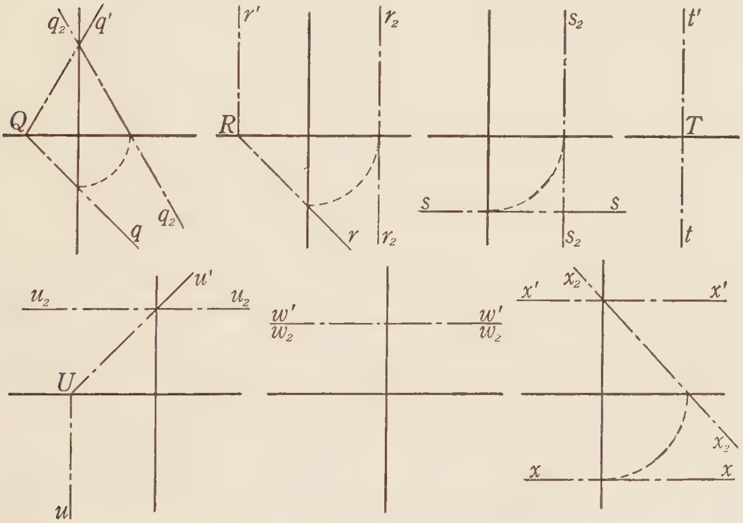

FIG. 37. Profile traces of planes

The student should determine and describe for himself the profile trace of each of these characteristic planes. This is done by means of the relations established in the preceding article and illustrated for each characteristic plane in Fig. 37.

These seven typical positions or directions of a plane taken together constitute what is often termed the *alphabet* of a plane. The term "alphabet" may be used even when the plane is confined to one quadrant upon definite expressed requirements, as is done in the case of some of the seven orthographic drawings of Fig. 36, or it may be used without reference to the extent or quadrant of the plane, the latter usage being preferred. The location of the profile plane has no effect upon the alphabet characteristics of a plane.

28. Projecting planes

In the foregoing summary of the relation of planes to the coördinate planes, numbers 2, 3, and 4 specify planes that are perpendicular to the horizontal plane. These planes are called H projecting planes for the obvious reason that the projection upon the H plane of a point, line, or plane geometrical figure lying in any one of them is coincident with the H trace of the projecting plane. In other words, such planes *project* anything that may be contained in them upon the H plane in a very real sense. Therefore the H trace of such a plane represents the H projections of innumerable lines in the projecting plane itself.

Likewise the planes of numbers 4, 5, and 6 of the summary are called V projecting planes, and the planes of numbers 3, 6, and 7 are called P projecting planes, for reasons similar to those indicated above for H projecting planes.

When an auxiliary coördinate plane is set up perpendicular to an oblique plane, as is done in Fig. 39, the oblique plane becomes a projecting plane with reference to the auxiliary plane.

Enumeration of the important uses of projecting planes must be left until later, but mention of the fact that these planes are much like auxiliary coördinate planes in their use, though represented by a different convention, will serve to emphasize their importance.

29. Parallel planes

If any two planes are parallel, their respective traces are parallel; and, conversely, if all the corresponding traces of any two planes are parallel, the planes are parallel. This proposition needs no formal proof since it will be remembered from solid geometry that when two parallel planes are cut by a third, the lines of intersection are parallel. In the present instance, the H, V, P, or an auxiliary coördinate plane is the "third plane" and the traces of the two parallel planes are the "lines of intersection." Fig. 38 will help the student to recall and apply these fundamental facts.

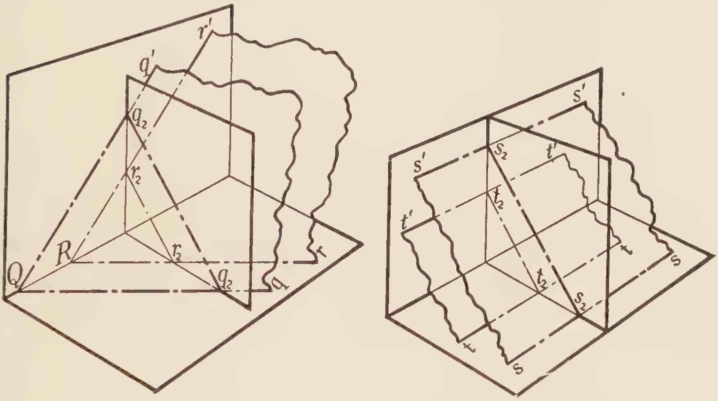

FIG. 38. Traces of parallel planes

As pointed out in the case of parallel lines, there will appear to be an exception to the general rule when the planes are perpendicular to the profile plane if only the H and V traces are considered. In the second part of Fig. 38 there are shown two parallel planes perpendicular to the profile plane. In this case it is obvious that the third set of traces is necessary in order to know definitely that the planes are parallel. In fact, the profile traces of such planes must be drawn before the H and V traces can be finally determined. In no other case are the profile traces necessary to determine the parallelism of two planes.

30. Traces of general planes on auxiliary coördinate planes

The trace of any plane on an auxiliary coördinate plane is obtained by the method shown in Fig. 39. In explanation of the construction involved, it is pointed out again that the trace of a general plane on any coördinate plane is simply the line of intersection of the given plane with that coördinate plane. In Fig. 39 the auxiliary plane 3 has been set up perpendicular to the H plane, and the line of intersection of plane T with plane 3 has been represented both as a trace, t_3t_3, and as a segment of a line, HV, in the first and second drawings respectively. The points H and V are in reality the

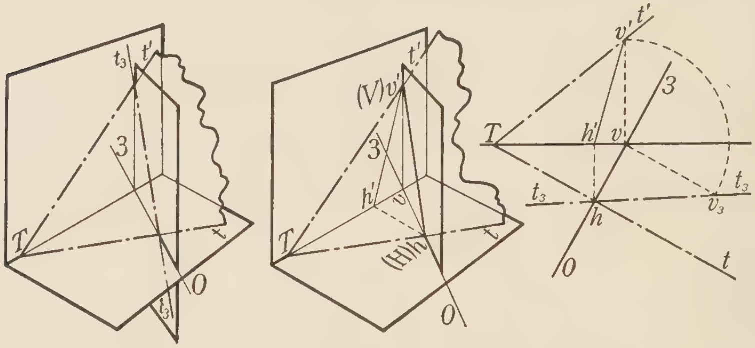

FIG. 39. Traces on auxiliary coördinate plane 3

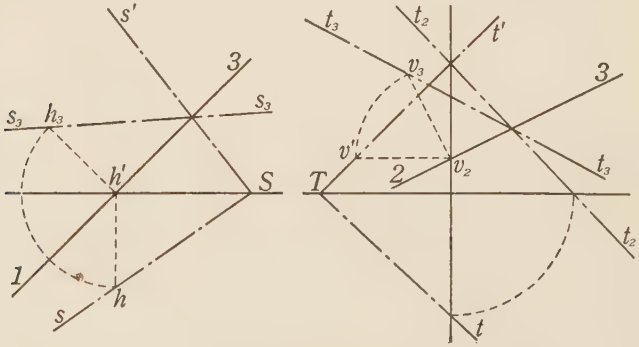

FIG. 40. Traces on auxiliary coördinate plane 3

piercing points of the trace t_3t_3 in the H and V planes respectively. When the auxiliary plane is revolved about its ground line into the H plane, the point H does not move, and the point V falls on a perpendicular to *0–3* through *v* and at a distance from *v* equal to the height of the point V above the ground line. This construction is carried out in orthographic projection in the last drawing of the figure, and the auxiliary trace, t_3t_3, is drawn through the two points thus determined.

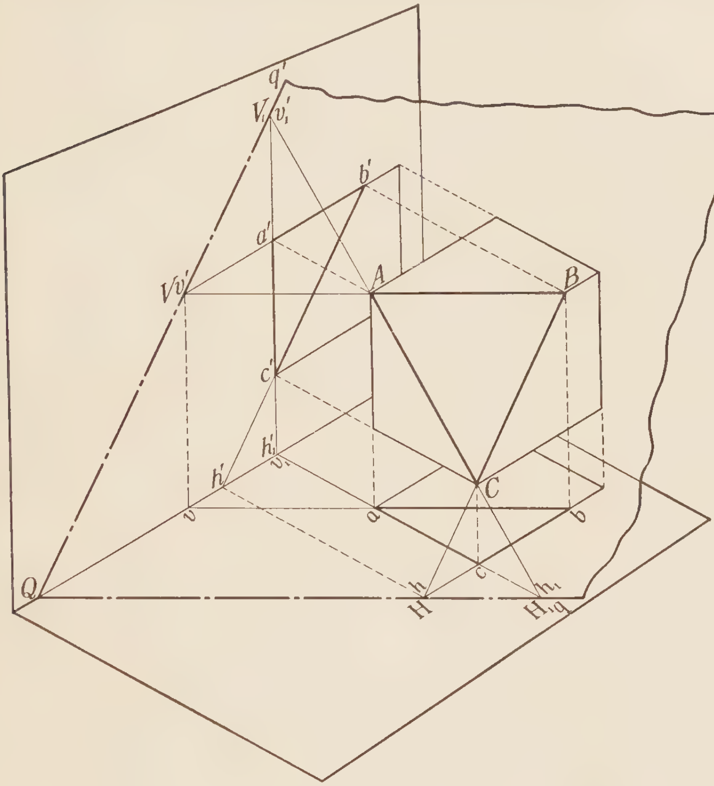

FIG. 41. Trace versus line representation of planes

In Fig. 40 the construction is shown for finding the auxiliary trace of a plane when plane 3 is set up perpendicular to either the V or the P plane.

31. Comparison of trace and line methods of representing planes

It is now possible to compare the two methods of representing indefinite planes and plane figures and to point out the connections between them. In Fig. 41 the upper left front corner of the block has been cut away by the plane Q. The triangle ABC, thus formed, is properly represented by the two triangular line projections, *abc* and *a′b′c′*, as shown. The indefinite plane of the triangle ABC is represented by the traces *Qq* and *Qq′*. The two types of representation of a plane surface are connected intimately by the fact that the lines of the triangle, since they lie in the plane Q, pierce the coördinate planes in the traces of the plane Q, as the figure illustrates. This fact is of the utmost importance in the development of the subject matter to follow.

CHAPTER IV

FUNDAMENTAL RELATIONS BETWEEN POINTS, LINES, AND PLANES

32. Problem procedure

The discussions of the three preceding chapters have been chiefly for the purpose of developing the methods of representing points, lines, and planes on the various coördinate planes. The system of nomenclature to be used in distinguishing between the several magnitudes with which descriptive geometry deals, and also between their representations on the coördinate planes, has been developed to a considerable extent. In the discussions to follow in this chapter, the fundamental relations which are continually met with in handling the point, line, and plane, both in the development of descriptive geometry and in practical engineering work, are presented in the form of problems to be analyzed and solved. The understanding acquirement of these fundamental relations and the principles upon which they depend will constitute the "working tools" of the student in his further progress.

In attempting to assimilate and understand what is to follow, the reader must bear in mind that the problem and its solution are space affairs; they are matters within both the realm of imagination and that of logical reasoning. The constructions on paper must also be logical in arrangement and sequence. They are carried on with projections and traces as the mediums of expression and are simply multiview representations of the space conditions. In most cases the following arrangement will be adhered to in the exposition and solution of any problem:

1. A statement of the problem to be solved.
2. An analysis of the problem into its component parts, including a statement of each principle, fact, or relation to be used in effecting a solution.
3. A chronological statement of the actual steps of construction in setting up the problem and in obtaining its solution.
4. A discussion of special conditions, if they are important.

33. To assume a point or an inclined line in a given plane

In Article 26 it was shown that the point of intersection of the H and V traces of a plane on the horizontal ground line is a point in the plane. This point is shown as O in the plane T in the first drawing of Fig. 42. It is the easiest of all the points in the plane to assume. It is not, however, the most advantageous point to use.

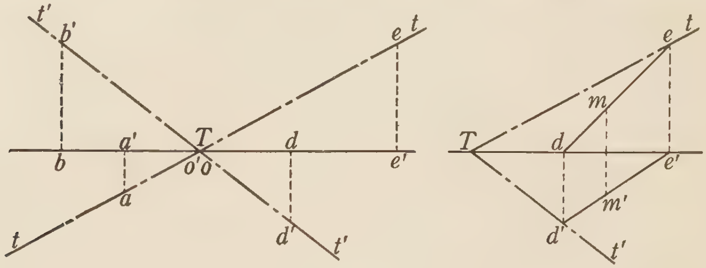

FIG. 42. Points in planes

The traces of planes are, of course, composed of innumerable points, any one of which may readily be shown in projection, as is done for B, A, D, and E of Fig. 42. Since these points are in either the H or the V plane, one projection of each must be in the horizontal ground line. This will be the vertical projection for points like A and E, since they are in the H plane; it will be the horizontal projection for points like B and D, since they are in the V plane.

In the second drawing of Fig. 42 the points D and E have been connected by a straight line. The line DE must lie, for its entire extent, in the plane T, since both its ends are in the plane; therefore any point chosen at random on this line, such as M, must lie in the plane T. The point M is any point in space in the plane T and not in a coördinate plane. It is the point most advantageously chosen for general work. By joining other chosen points on the traces of the plane T lines may be secured in any quadrant and pointing in any direction in the plane, thus making it possible to select a point on a line in any specified position.

Fig. 43 illustrates the general inclined line in all four quadrants, but always in the plane T, as is shown by the fact that the ends of the lines are in the traces of plane T. Reference to Article 18 will serve to recall to the student that the ends of these lines are really the H and V piercing points of the lines and should, therefore, be lettered H and V, a procedure strictly adhered to in the remaining portions of this text. Further, it must be understood that a point H and its horizontal projection, *h*, coincide. The letter H is not put on the drawing, since piercing points are always represented by their projections, like other points in space. Similarly, a point V coincides with its vertical projection, v', and point P coincides with p_2. The point P is often represented by its H and V projections only.

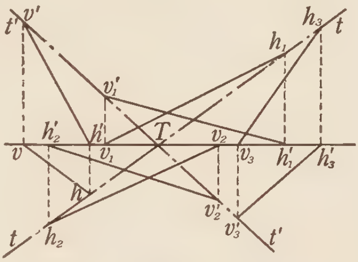

FIG. 43. Lines in a plane

34. To assume a line parallel to any coördinate plane and in a given plane

In Fig. 44 the effect of varying the inclination of a line in a plane, by holding one of its piercing points fixed and moving the other along the corresponding trace, is shown. The point V, in the first drawing of the figure, is kept stationary while the point H is moved to successive positions H_1, H_2, etc. on *Rr*. Finally, the H piercing point is moved to infinity on the H trace. The H projection of the line becomes parallel to the H trace,* and the V projection becomes parallel to the horizontal ground line. Only a segment, VA, of this line can be lettered. The line VA is parallel to the H plane and is in the plane R; therefore it is said to be an H-parallel of (in) the plane R. The position of the line VA in space and the relations described in this paragraph, as well as those of the succeeding paragraph, may be more clearly understood with the aid of the pictorial sketches directly below the orthographic drawings of the figure.

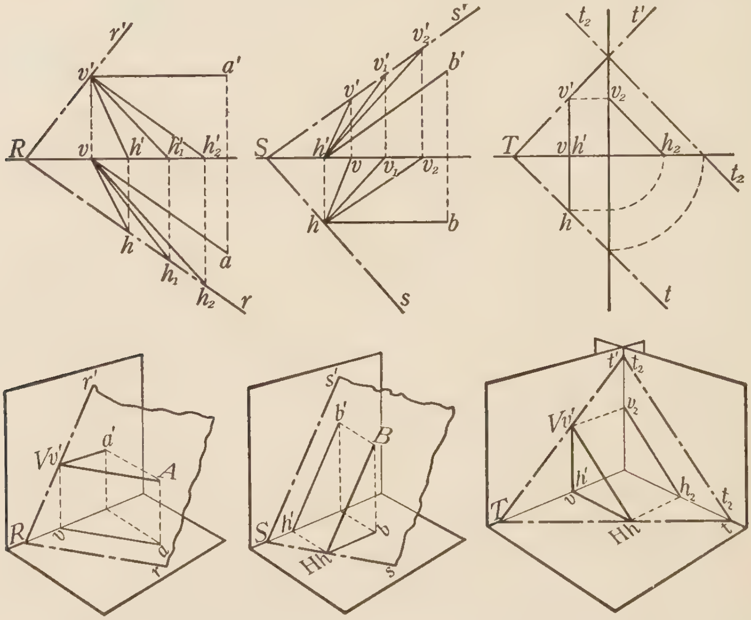

FIG. 44. H-, V-, and P-parallels of an oblique plane

In the second drawing of Fig. 44 the H piercing point remains fixed and the V piercing point is moved along the

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* Parallel lines are assumed to meet at infinity in considering the higher geometrical relations.

V trace of the plane S to the successive positions V_1, V_2, etc., and finally to infinity. The line HB results from this manipulation. Its V projection is parallel to the V trace of plane S, and its H projection is parallel to the horizontal ground line. It is a V-parallel of (in) the plane S.

In the third drawing of Fig. 44 the points V and H are so chosen in the traces of the plane T that the line VH is a P-parallel of (in) that plane. Its H and V projections are perpendicular to the horizontal ground line, and its profile projection is parallel to the profile trace of the plane.

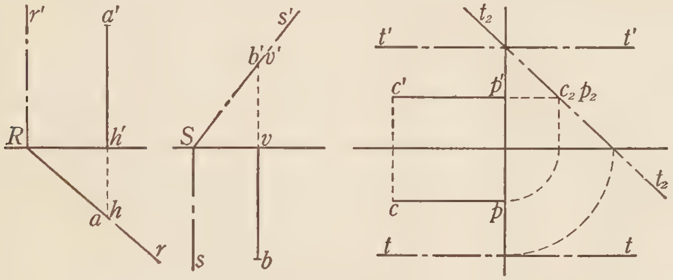


FIG. 45. H-, V-, and P-parallels of projecting planes

If the given plane is a projecting plane (see Article 28) and the piercing point on the inclined trace is kept fixed while the other piercing point is moved to infinity along the corresponding trace, then the types of lines shown in Fig. 45 are obtained. The three lines illustrated in the figure are not only coördinate-plane-parallels, but they are also perpendicular to one or another of the three principal coördinate planes and hence have one of their projections a point.

In Fig. 46 the line HV has been projected upon plane 3, to which it is parallel. The trace of the plane T on the auxiliary plane 3 is t_3t_3, parallel to h_3v_3. Since HV is in plane T and parallel to plane 3, it becomes a 3-parallel of (in) plane T. It will be noticed that the horizontal and auxiliary projections and traces, together with the ground line *0–3*, are arranged in respect to each other in identically the same way as are the projections, traces, and ground lines in the case of each plane-parallel of Fig. 44; in fact, the whole drawing may be turned so that *0–3* shall appear to be the horizontal ground line. The same constructions may be performed with *0–3* as the ground line that are possible with the original ground line as the line of reference.

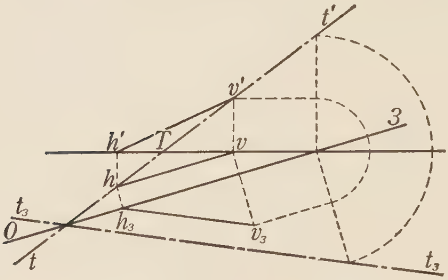


FIG. 46. A 3-parallel of a plane

The several coördinate-plane-parallels of Figs. 44, 45, and 46 are very important lines in the solution of problems. The following summary concerning them will aid in fixing in mind the characteristics of this most important group of lines. At least three coördinate planes must be used to make the statements in the summary perfectly general and applicable to every line and plane in space.

1. A plane-parallel of (in) a given plane has at least one projection parallel to a ground line; another projection is parallel to a corresponding trace of the given plane.
2. One piercing point is at infinity on the corresponding trace of the given plane; another piercing point is in the corresponding trace of the given plane and at a finite distance from the intersection of the trace with a ground line.

35. To pass a plane through a given line which is inclined to the H, V, and P planes

Analysis. Any number of planes may be passed through a line which is inclined to the coördinate planes. The traces of every plane passed through such a line must contain the corresponding piercing points of the line and meet on the proper ground lines.

Construction. Given the line AB, Fig. 47. Produce AB until it pierces the coördinate planes at the points V, H, and P. Draw the V trace, Tt', through v' and intersecting the horizontal ground line at any point T. Draw the H trace, Tt, through h and intersecting the ground line at T. Draw t_2t_2 through p_2, intersecting Tt' on the vertical ground line. Plane T is the required plane.

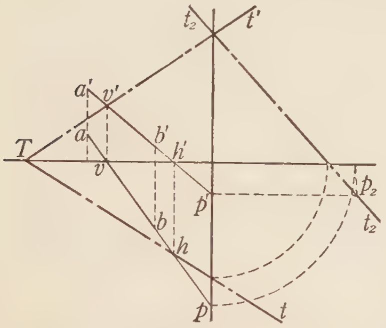


FIG. 47. General plane through an inclined line

Fig. 48 indicates other characteristic planes which might have been selected to pass through the line AB instead of the plane T as shown in Fig. 47. They are projecting planes in each instance. It should be observed that planes R and S can be passed through the line without even finding its piercing points, because one trace must contain the corresponding projection of the line and the other trace must be perpendicular to the ground line.

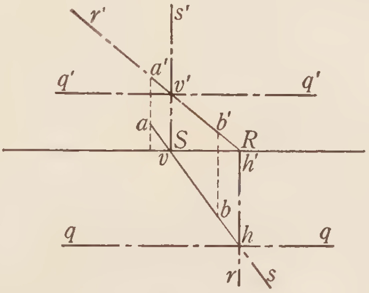


FIG. 48. Projecting planes through an inclined line

NOTE. It is customary to require only the H and V representations in the solution of problems. Hereafter, except when the solution cannot be obtained readily without other representations, the constructions will be confined to the H and V planes. When the projection of a line coincides with the trace of a plane, the drafting technique shown with Rr' and $a'b'$, Fig. 48, is generally followed. This is in conformity with the general rule that a full-line projection shall take precedence over a broken-line representation. Care should be taken to show the exact point of change from one line to another.

36. To pass a plane through a given line which is parallel to a coördinate plane

Analysis. Any number of planes may be passed through a line which is parallel to a coördinate plane. Such a line has one piercing point at infinity. Any plane containing the line must have one trace passing through the corresponding available piercing point and the other trace parallel to the corresponding projection of the line.

Construction One. Given the H-parallel AB, Fig. 49. Produce the line until it pierces the V plane at the point V.

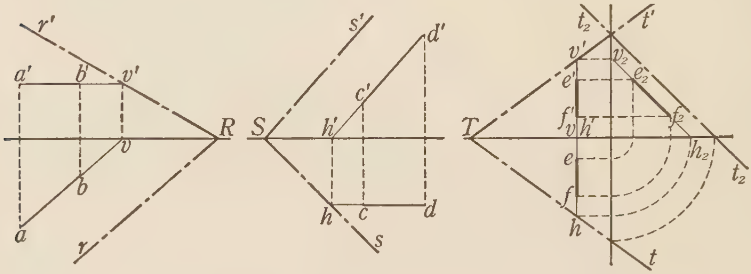


FIG. 49. Planes through lines that are H-, V-, and P-parallels

Draw the V trace, Rr', of the required plane through v' in any desired direction and intersecting the ground line at the point R. Draw the H trace, Rr, through the point R, parallel to ab (see Fig. 44). Plane R is the required plane.

Construction Two. Given the V-parallel CD, Fig. 49. Produce the line until it pierces the H plane at the point H. Draw the H trace, Ss, of the required plane through h in any desired direction and intersecting the ground line at the point S. Draw Ss' through the point S parallel to $c'd'$ (see Fig. 44). Plane S is the required plane.

Construction Three. Given the P-parallel EF, Fig. 49. Produce the line EF until it pierces the V plane at the point V and the H plane at the point H. This is done by first projecting EF upon the profile plane at e_2f_2 and extending e_2f_2 to the vertical and horizontal ground lines to locate v_2 and

h_2 respectively. Draw the V trace, Tt', of the required plane through v' in any desired direction and intersecting the horizontal ground line at the point T. Draw Tt through the points T and h. Draw t_2t_2, if needed, parallel to e_2f_2 and through the point of intersection of Tt' with the vertical ground line (see Fig. 44). Plane T is the required plane.

37. To pass a plane through two intersecting lines

Analysis. Only one plane may be passed through two intersecting lines. The H and V traces of the required plane must contain the corresponding piercing points of each line, and they must intersect on the horizontal ground line.

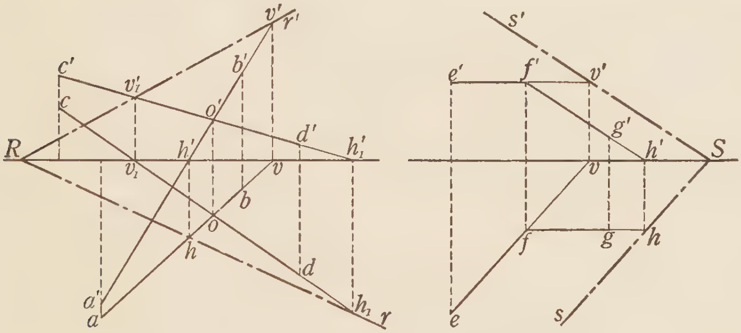


FIG. 50. Planes through intersecting lines

Construction One. Given the lines AB and CD, intersecting at the point O, in the first drawing of Fig. 50. Produce the two lines to pierce the V plane at the points V and V_1 respectively; then produce the lines to pierce the H plane at the points H and H_1 respectively. Draw Rr' through v' and v_1' and intersecting the ground line at the point R. Draw Rr through h and h_1. Plane R is the required plane.

Construction Two. Given the lines EF and FG, intersecting at the point F, in the second drawing of Fig. 50. Produce EF, an H-parallel, to pierce the V plane at the point V. Produce FG, a V-parallel, to pierce the H plane at the point H. No other H or V piercing points are available. Draw *Ss'* through *v'* parallel to *f'g'* to intersect the ground line at the point *S* (see Fig. 44). Draw *Ss* through *h* parallel to *fe*. Plane S is the required plane.

NOTE. Always, in drawing the projections of lines and the traces of planes, two points on the projection or trace must be determined or one point and the direction must be known, in order to draw either representation. Direction is indicated by stating that the projection or trace is parallel or perpendicular to some other known line on the drawing or by stating the degree of inclination it shall have with another line. Wherever one of two or more intersecting lines is a plane-parallel, then the direction of one trace of the plane containing these lines is known, namely, parallel to that projection of the plane-parallel that is inclined to the ground line.

Construction Three. Given the lines KM and NP, intersecting at the point O, Fig. 51. The line KM is nearly parallel to the H and V planes; hence its H and V piercing points are outside of the limits of the drawing. Connect K and N by the straight line KN, as shown. The line KN lies in the required plane by the definition of a plane (see Article 24). Produce the line NP to obtain its H and V piercing points. Produce KN to pierce the coördinate planes at points H_1 and V_1. Draw *t't'* through *v'* and v_1'. Draw *tt* through *h* and h_1. Plane T is the required plane.

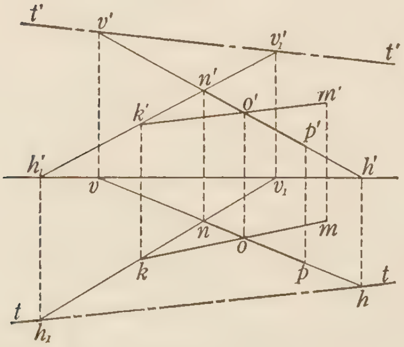


FIG. 51. Plane through intersecting lines

Discussion. The traces of the plane T of Fig. 51 do not intersect on the ground line within the limits of the drawing, a result which was anticipated from the inclination of the line KM. Had KM been parallel to the H and V coördinate planes, — both projections parallel to the ground line, — then the plane T would have been perpendicular to the profile plane and its traces would have been parallel to the corresponding projections of KM. In such a case the auxiliary line KN could have been omitted, because the direction of each trace and a point on it would have been known.

The method of drawing an auxiliary (helping) line which was employed in Construction Three to determine the plane of two intersecting lines is also applicable to two parallel lines or to plane geometrical figures such as the triangle, parallelogram, circle, ellipse, or other plane curves. The points to be joined must be selected with care in order that the piercing points of the resulting lines may be within the limits of the drawing. Fig. 52 illustrates the use of these auxiliary lines in a few of the more important cases mentioned above.

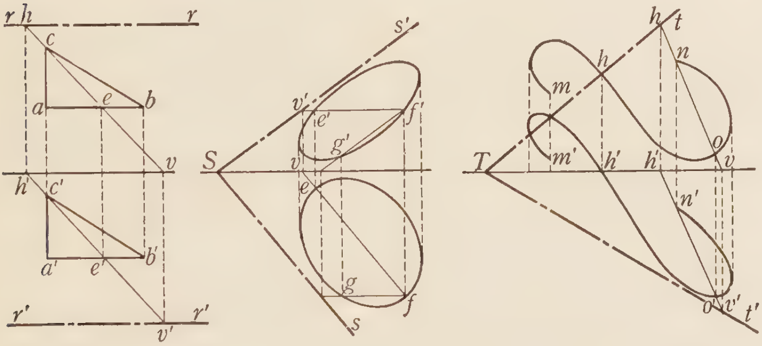


FIG. 52. Planes through plane geometrical figures

In each of the constructions of Figs. 50 and 52 the point of intersection of the H and V traces was determined as a check on the construction. It might have been used as a construction point, when available on the drawing, and one or another of the piercing points omitted. This will be done, hereafter, in many of the constructions.

38. To pass a plane through two given parallel lines

Analysis. The traces of the required plane must contain the respective piercing points of each line and intersect on the ground line.

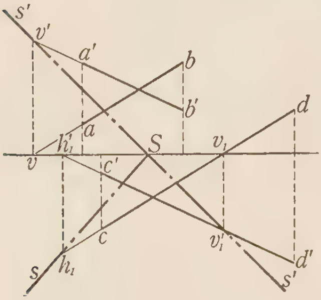


FIG. 53. Plane through parallel lines

Construction. Given the parallel lines AB and CD, Fig. 53. Produce the lines AB and CD to obtain any three of their four piercing points. The line AB pierces the V plane at the point V. The line CD pierces the V plane at the point V_1. It pierces the H plane at the point H_1. Draw $s's'$ through v' and v_1' to intersect the ground line at the point S. Draw Ss through h_1 and the point S. Plane S is the required plane.

39. To pass a plane through a given point and line

Analysis. If the given point is joined to some chosen point on the given line by a straight line, or if a line is drawn through the point parallel to the given line, the plane of the two intersecting or parallel lines will be the required plane.

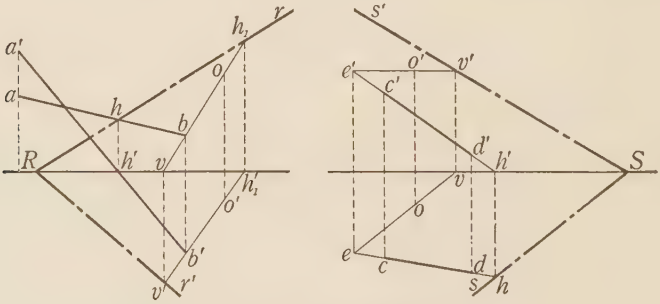


FIG. 54. Plane through point and line

Construction One. Given the point O and the line AB in the first drawing of Fig. 54. Connect the point O to the point B by the line OB, the projections of which are *o′b′* and *ob*. Produce the lines AB and OB to pierce the H plane at H and H_1 respectively. Produce OB to pierce the V plane at V. Draw *Rr* through *h* and h_1 to intersect the ground line at the point *R*. Draw *Rr′* through *v′* and point *R*. Plane R is the required plane.

Construction Two. Given the point O and the line CD in the second drawing of Fig. 54. Connect the point O to the point E on the line CD produced, so that *o′e′* is parallel to the ground line. The line OE is an H-parallel. Find the V piercing point of OE and the H piercing point of CD. Draw the trace *Ss* through *h* parallel to *oe*. Draw *Ss′* through *v′* and the point *S*. Plane S is the required plane.

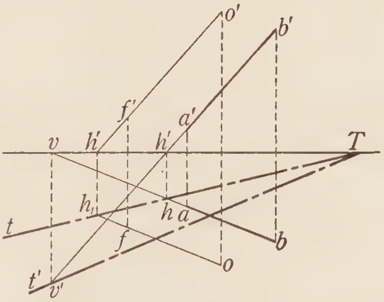


FIG. 55. Plane through point and line

Construction Three. Given the point O and the line AB, Fig. 55. Draw the line OF parallel to the line AB. The projections are *o′f′* and *of*. Produce the lines AB and OF to pierce the H and V planes at the three piercing points H, V, and H_1. Draw *Tt* through *h* and h_1 to intersect the ground line at the point *T*. Draw *Tt′* through *v′* and the point *T*. Plane T is the required plane.

40. To pass a plane through three given points

Analysis. If one of the given points is connected to the other two, two intersecting lines are obtained. The plane of these two lines is the required plane.

Construction One. Given the three points A, B, and C in the first drawing of Fig. 56. Connect the point B to the points A and C to obtain the intersecting lines AB and BC. The H and V piercing points of these two lines are H and H_1, V and V_1, respectively. Draw *Rr* through h and h_1. Draw *Rr′* through v' and v_1'. Plane R is the required plane.

Construction Two. Given the points D, E, and F in the second drawing of Fig. 56. Connect the point D to the point E by a straight line, and draw the line FG so that the point G is on the line DE and *f′g′* is parallel to the ground line.

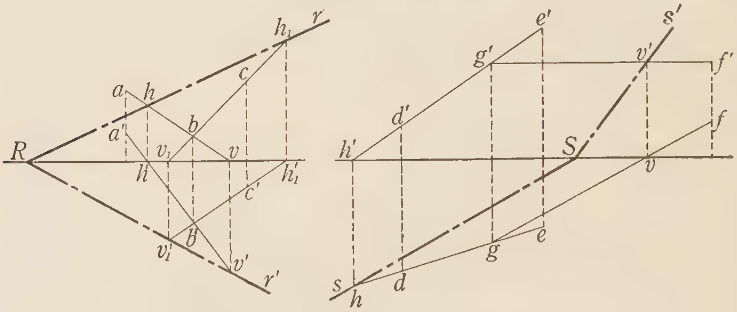


FIG. 56. Plane through three points

Find the H piercing point of the line DE and the V piercing point of the line FG. Since the line FG is an H-parallel, *Ss* is drawn parallel to *fg* through h; *Ss′* is drawn through v' and the point *S*. Plane S is the required plane.

41. To pass a plane through a given point and parallel to a given line

Analysis. Any number of planes may be passed through a point parallel to a given line. If a line is drawn through the given point parallel to the given line, any plane containing this line will be the required plane.

42. To pass a plane through one given line and parallel to another given line

Analysis. If from some chosen point on the one line an auxiliary line is drawn parallel to the second given line, the plane of the two intersecting lines will be the required plane.

43. To pass a plane through a given point and parallel to two given lines

Analysis. The required plane must contain two lines drawn through the given point parallel to the two given lines.

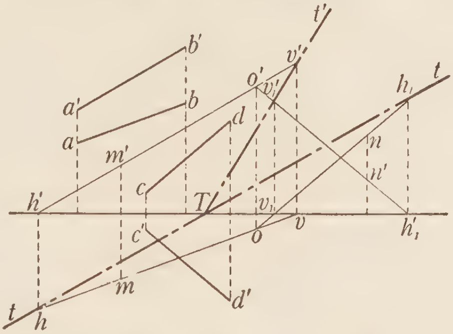


FIG. 57. Plane through a point parallel to two lines

Construction. Given the point O and the lines AB and CD, Fig. 57. Through the point O draw the line OM parallel to AB and the line ON parallel to CD. Produce OM and ON to pierce the H and V planes at the points H and H_1, V and V_1, respectively. Draw *tt* through *h* and h_1 to intersect the ground line at the point *T*. Draw *Tt'* through *v'* and v_1'. Plane T is the required plane.

44. To pass a plane through a given point and parallel to a given plane

Analysis. Only one plane may be passed through a given point parallel to a given plane. The required plane must have its traces parallel to the corresponding traces of the given plane (see Article 29). It must contain two lines drawn through the given point parallel to any two lines of the given plane.

Construction One. Given the point O and the plane S in the first drawing of Fig. 58. Through the point O draw the lines OA and OB parallel, respectively, to imagined H- and V-parallels of the plane S. The two available piercing points of the lines OA and OB are V and H respectively. Draw *Rr'* through *v'* parallel to *Ss'* or *o'b'*, and draw *Rr* through *h* parallel to *Ss* or *oa*. Plane R is the required plane.

Construction Two. Because of the fact that the traces of the required plane must be parallel to the corresponding traces of the given plane, only one of the lines of Construction One of Fig. 58 need be drawn to obtain a solution. In the second part of the figure the point O and the plane Q are given. Draw OC parallel to an imagined H-parallel of the plane Q. The H trace, *Qq*, may be considered an H-parallel of the plane Q for the purpose of making this construction; its V projection is in the ground line. Produce OC to pierce the V plane at the point V. Draw *Tt′* through *v′* parallel to *Qq′*. Draw *Tt* through *T* parallel to *Qq* and *oc*. Plane T is the required plane.

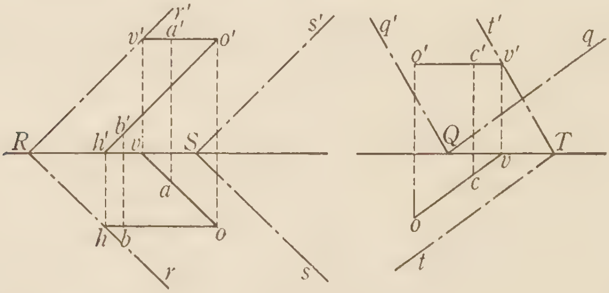


FIG. 58. Plane through a point parallel to a plane

Discussion. If the given plane is a projecting plane, the required plane may be passed directly without the intermediate step of drawing a line through the given point.

45. Lines perpendicular to planes

Thus far the fundamental relations between lines and planes have been confined to those which may be expressed by means of the phrases "inclined to" or "parallel to." Another important relation between a line and a plane is that of perpendicularity. When a line is perpendicular to a plane, its projections are perpendicular to the corresponding traces of the plane and, conversely, if the projections of a line are perpendicular to the corresponding traces of a plane, the line is perpendicular to the plane. The usefulness of this relation cannot be overstated.

The proof of the relation cited in the previous paragraph may be visualized with the aid of the first drawing of Fig. 59. The line AP is perpendicular to the plane T. Through the line AP a plane R is passed perpendicular to the H plane. Plane R is the H projecting plane of the line AP and is also perpendicular to the plane T, since it contains the line AP, which is perpendicular to the plane T. The plane R, being perpendicular to both planes T and H, must be perpendicular to *Tt*, their line of intersection. *Tt* is the H trace of the plane T and is perpendicular to all lines in plane R passing through point *M*, the point of intersection of the H traces of planes R and T. The line (projection) *ap* is in the plane R, passes through the point *M*, and is at the same time the H projection of the line AP. Therefore *ap* is perpendicular to *Tt*.

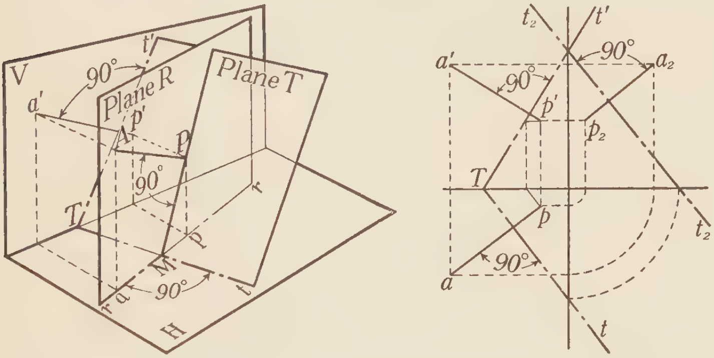


FIG. 59. Projections of a line perpendicular to a plane

By the use of the V projecting plane of the line AP, it can be shown that the V projection of AP and the V trace of plane T are also perpendicular. Similarly, it can be shown that the projection of AP on any coördinate plane will be perpendicular to the trace of plane T on that plane. The second drawing of Fig. 59 shows these relations in the usual orthographic projections.

In drawing a line through some point O in space perpendicular to a P projecting plane, such as the plane S in the first drawing of Fig. 60, or in passing a plane through a point O in space perpendicular to a line which is a P-parallel, such as the line CD in the second drawing of Fig. 60, it is almost always necessary to apply the principle just explained on a third plane of projection before the H and V representations may be determined. This is so in the first instance because any number of lines can be drawn through a point with their H and V projections perpendicular to the corresponding traces of a P projecting plane, yet there can be only one line through a given point perpendicular to such a plane. Similarly, in the second instance any number of planes may be passed through a given point with their H and V traces perpendicular to the corresponding projections of a P-parallel, yet only one plane can be passed through the point perpendicular to the line. In the figure the profile plane was used as the third coördinate plane, although any auxiliary plane would have served the purpose.

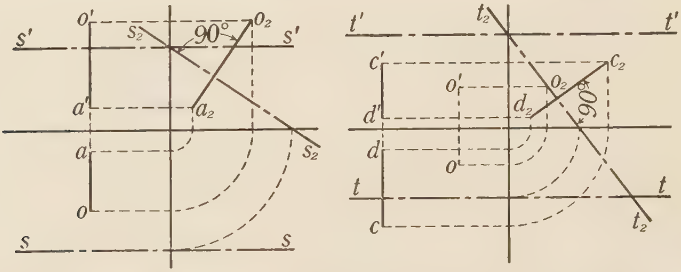


FIG. 60. Projections of a P-parallel perpendicular to a P projecting plane

46. Parallel and perpendicular relations of lines and planes

It is now possible to state in summary form the relations of parallelism and perpendicularity between two lines, two planes, or a line and a plane, in terms of the corresponding relations between their projections or traces, as follows:

1. If two lines are parallel, their corresponding projections are parallel; and, conversely, if the corresponding projections of two lines are parallel, the lines are parallel (see Article 21).
2. If two planes are parallel, their corresponding traces are parallel; and, conversely, if the corresponding traces of two planes are parallel, the planes are parallel (see Article 29).
3. If a line is perpendicular to a plane, its projections are perpendicular to the corresponding traces of the plane; and, conversely, if the projections of a line are perpendicular to the corresponding traces of a plane, the line is perpendicular to the plane (see Article 45).
4. If two lines are perpendicular, their corresponding projections cannot be perpendicular.
5. If two planes are perpendicular, their corresponding traces cannot be perpendicular.
6. If a line is parallel to a plane, its projections cannot be parallel to the corresponding traces of the plane.

The first three statements of relations in the preceding summary are positive in character and may be utilized in the solution of problems; the last three statements are negative in character and are useful only in preventing erroneous solutions. In both groups it is necessary, in special cases, to apply the rules to more than two coördinate planes, as was done in Fig. 60, in order to eliminate apparent exceptions to their universal applicability.

47. To pass a plane through a given point and perpendicular to a given plane

Analysis. Any number of planes may be passed through a point perpendicular to a given plane. Any one of these planes will contain a line drawn through the given point and perpendicular to the given plane.

Construction. Given the point O and the plane T, Fig. 61. Draw OA perpendicular to the plane T; *o′a′* is perpendicular to *Tt′*, and *oa* is perpendicular to *Tt*. Any one of the planes Q, R, S, and U, in the first and second drawings of the figure, passed through OA at various inclinations to the H and V planes, is the required plane.

Discussion. In Fig. 61 plane R is perpendicular to the horizontal plane and plane U is perpendicular to the vertical plane. These planes may be transformed into auxiliary coördinate planes by merely changing the traces *Rr* and *Uu'* to ground lines and omitting the other two traces altogether. This is what has been done in the third part of the figure, where an auxiliary coördinate plane 3, represented by the ground line *0–3*, has been set up perpendicular to the H plane, and where another auxiliary coördinate plane 3, represented by the ground line *1–3*, has been set up perpendicular to the V plane, each auxiliary plane being passed through a chosen point O. The P projecting plane S can be treated similarly to obtain a new ground line on the P plane. The constructions of this article may be utilized, therefore, in setting up auxiliary coördinate planes perpendicular to other planes, giving the relation mentioned in Article 28.

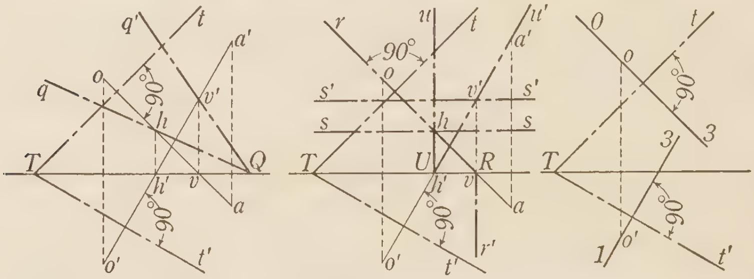


FIG. 61. Plane through a point perpendicular to a plane

48. To pass a plane through a given line and perpendicular to a given plane

Analysis. If from some chosen point on the given line an auxiliary line is drawn perpendicular to the given plane, the plane of the two intersecting lines will be the required plane.

Construction. Given the line AB and the plane T, Fig. 62. Draw the line BC perpendicular to the plane T; *b′c′* is perpendicular to *Tt′*, and *bc* is perpendicular to *Tt*. Produce the lines AB and BC to intersect the H and V planes at H, V, and V_1. Draw *Ss′* through *v′* and v_1' to intersect the ground line at the point *S*. Draw *Ss* through the points *S* and *h*. Plane S is the required plane.

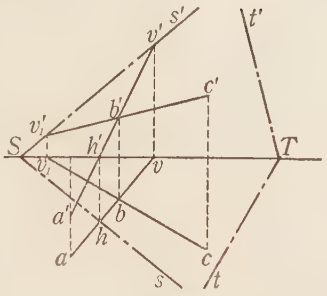


FIG. 62. Plane through a line perpendicular to a plane

49. To pass a plane through a given point and perpendicular to two given planes

Analysis. If two lines are drawn through the given point so that one of them is perpendicular to the first given plane and the other is perpendicular to the second given plane, the plane of these two intersecting lines is the required plane.

Construction. The steps of construction here are like those of the preceding article except that two lines perpendicular to the planes are drawn.

50. To pass a plane through a given point and perpendicular to a given line

Analysis. Any plane passed perpendicular to a line will have its traces perpendicular to the corresponding projections of the line (see Article 45). If such a plane is passed at random, then a plane passed parallel to it (see Article 44) through the given point will be the required plane.

Construction. Given the point O and the line AB, Fig. 63. Pass a plane T at random, but perpendicular to the line AB; *Tt′* is perpendicular to *a′b′*, and *Tt* is perpendicular to *ab*. Draw the V-parallel, OH, through the point O and parallel to an imagined V-parallel in the plane T. The vertical projection, $o'h'$, must be parallel to Tt', and the H projection, oh, must be parallel to the ground line. Pass the plane S through OH parallel to the plane T by drawing Ss through h parallel to Tt and by drawing Ss' through the point S parallel to Tt'. Plane S is the required plane.

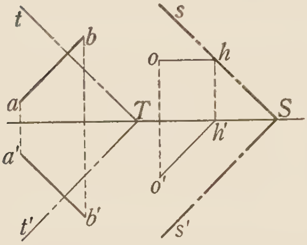


FIG. 63. Plane through a point perpendicular to a line

Discussion. The line OH can be drawn without the aid of the plane T, since its vertical projection, being parallel to Tt', must be perpendicular to $a'b'$, and the traces of the plane S may be drawn without the aid of the traces of the plane T, since each trace of the plane S must be perpendicular to the corresponding projection of AB; hence it is customary to omit the plane T altogether in solutions of this problem.

When the given line is an H-, V-, or P-parallel, the required plane will be an H, V, or P projecting plane, respectively, and may be passed directly without the aid of an intermediate line.

51. To find the line of intersection of two given planes

Analysis. By definition, the line of intersection of any two planes is a line that is common to the planes; in other words, it is the locus of points in the one plane that also lie in the other. Any two of these locus points, or any one of them and a stated direction, will determine the required line of intersection. Since the H and V piercing points of every line in the one plane lie in the respective H or V trace of that plane, and since the H and V piercing points of every line in the second plane lie in the respective H or V trace of that plane, the intersections of the corresponding traces of the two planes must be the respective H and V piercing points of the required line of intersection.

These two points may be used only when available on the drawing or when their positions can be definitely stated in direction from some other point on the line of intersection that can be obtained; otherwise, two intermediate points on the line of intersection must be found by the method of Construction Five.

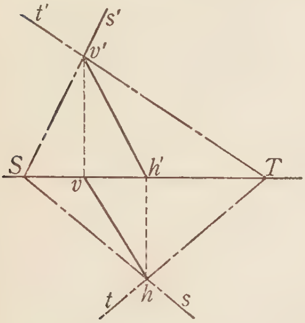


Fig. 64. Line of intersection of two planes

Construction One. Given the two intersecting planes S and T, Fig. 64. Produce *Ss'* and *Tt'* to intersect at the point V, whose H and V projections are *v* and *v'* respectively. Produce *Ss* and *Tt* to intersect at the point H with projections at *h* and *h'*. Join the points V and H by a straight line. The line VH is the required line of intersection of the planes S and T. It may be produced in either direction indefinitely as the intersecting planes are extended.

Construction Two. Given the two intersecting planes R and S, with their vertical traces parallel, Fig. 65. One of the points on the line of intersection is its H piercing point, obtained as in Construction One. The V piercing point of the line of intersection is at infinity, where the vertical traces of the planes meet. The projections of this point can only be indicated by direction arrows, as is shown in the figure. The projections of a segment, HA, of the infinite line, HV, can now be drawn, since a starting point and the direction

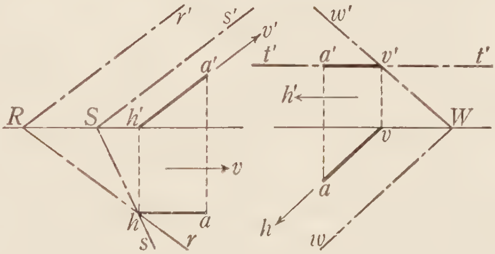


Fig. 65. Line of intersection of planes

of each projection are known. Draw $h'a'$ parallel to Ss' and Rr'. Draw ha parallel to the ground line. The point A is any chosen point on the infinite line HV. HA is the required line.

Construction Three. Given the two intersecting planes T and W, Fig. 65. The plane T is a V projecting plane; hence the vertical projection of the line of intersection must lie in $t't'$. Any line with its V projection parallel to the ground line must be an H-parallel, in this case, of the planes T and W; hence the other projection of the line of intersection must be parallel to the H trace of the plane W. Obtain v' and v as in Construction One. Draw $v'a'$, of any length, in $t't'$. Draw va parallel to Ww. The point A is any chosen point on the line VH with H at infinity. VA is the required line.

Construction Four. Given the two intersecting planes S and T, Fig. 66. Obtain the V and H piercing points of the line of intersection as in Construction One, — they are together upon the ground line, as shown, and so do not determine any definite line. This merely indicates that the line of intersection passes through the ground line at the coincident points H and V. Pass the plane R parallel to the plane S at some distance from it. Draw V_1H_1, the line of intersection of T and R. The required line of intersection of S and T must be parallel to V_1H_1, by solid geometry. Draw $v'a'$ parallel to $v_1'h_1'$. Draw va parallel to v_1h_1. VA (or HA) is the required line of intersection.

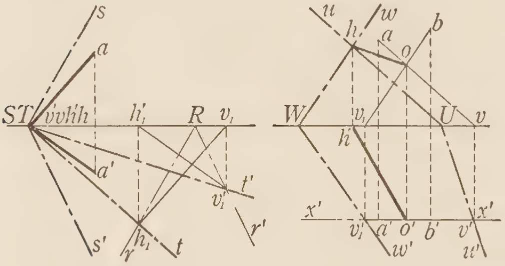


FIG. 66. Line of intersection of planes

Construction Five. Given the two intersecting planes U and W, Fig. 66. Obtain h and h' as in Construction One.

The V traces do not intersect within the limits of the drawing; hence the V piercing point of the line of intersection is not available. Pass the plane X parallel to the H plane, as shown in the figure. Find the lines of intersection of planes X and U and planes X and W, in turn, by the method of Construction Three. They are VA and V_1B respectively. VA and V_1B, being common to plane X, must intersect, which they do at the point O. The point O is in all three planes; hence it must be on the line of intersection of any two of them and therefore on the line of intersection of planes U and W. Join h to o and h' to o' to obtain HO, the required line of intersection.

Discussion. It should be noted that the methods of Constructions Four and Five may be applied to practically any arrangement of the two planes. The student must use his judgment and skill in selecting the proper construction to use. Some lines of intersection may be obtained by inspection and should be drawn without the aid of any of the methods demonstrated. Finding the line of intersection of two planes is a very common procedure in obtaining the solutions of more advanced problems. The methods here explained should be mastered by the student in every detail. The procedures to be followed when both planes are represented by definitely bounded plane figures are demonstrated in Articles 92, 93, and 94 of Chapter V.

52. To find the point in which a given line pierces a given plane

Analysis. The point in which a line pierces a given plane will lie in any plane which may be passed through the given line. Since the piercing point must lie in the given plane and in the plane passed through the given line, it will be found in the line of intersection of the two planes. Since the required point is in the given line and the line of intersection of the two planes, it will be at the intersection of these two lines.

If the given plane is a projecting plane, the piercing point may be found without passing a plane through the given line.

Construction One. Given the line AB and the plane S in the first drawing of Fig. 67. Pass the H projecting plane R through the line AB (Article 35). The line of intersection of planes S and R is VH, with projections *v'h'* and *vh*. The point of intersection of VH and AB produced is P, the required piercing point. Its projections are obtained by extending *a'b'* to intersect *v'h'* at *p'* and erecting a projecting line to intersect *ab* and *vh* at *p*.

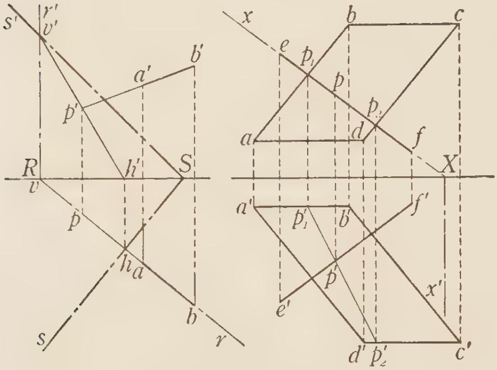


FIG. 67. Point in which a line pierces a plane

Construction Two. Given the line EF and the definite plane surface ABCD in the second drawing of Fig. 67. Pass the H projecting plane X through the line EF, as in the first step of Construction One. The surface ABCD may be represented by an unlimited oblique plane with definitely determined traces, and the solution may be obtained by following out the second and third steps of Construction One. However, since the line of intersection obtained in the second step of the preceding construction in the present case runs only from one side of the parallelogram to another, that is, from the piercing point of AB in plane X to the piercing point of CD in the same plane, it may be obtained without the use of the conventional plane through ABCD, as follows: Find the intersection of *ab* and *Xx* at p_1. Erect a vertical projecting line through p_1 to intersect *a'b'* at p_1'. P_1 is the piercing point of AB in plane X. In a similar manner locate p_2 and p_2'. P_2 is the piercing point of CD in plane X. Join P_1 and P_2 by a straight line. P_1P_2 is the line of intersection of the plane X and the plane surface ABCD, and corresponds to VH in Construction One. The given line EF intersects the line P_1P_2 at P, the required piercing point.

Construction Three. Given the line AB and the plane S, Fig. 68. Set up the auxiliary coördinate plane 3 perpendicular to the plane S and to the V plane (see Article 47). The projection of AB on plane 3 is a_3b_3, and s_3s_3 is the trace of plane S on plane 3; both are obtained by the usual constructions and are fully shown in the figure. Plane S becomes a projecting plane with reference to plane 3, its vertical trace being perpendicular to the ground line *1–3*; hence the piercing point of the line AB in the given plane may be located by means of the plane-3 view in the same manner as were P_1 and P_2 in Construction Two. The auxiliary trace, s_3s_3, is the edgewise view of plane S; therefore this trace will contain all auxiliary projections of points, lines, or other magnitudes which may be in plane S. Produce a_3b_3 until it intersects s_3s_3 to locate p_3. By means of the proper return projecting lines, p' and p are located on $a'b'$ and ab respectively. P is the required piercing point.

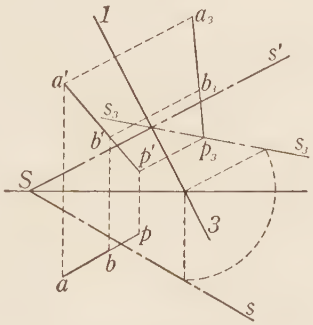


FIG. 68. Point in which a line pierces a plane

Discussion. The method of Construction Two is the practical or drafting-room method of obtaining the point in which a line pierces a plane, with lettering and ground line omitted, of course.

The short-cut construction for finding the point in which a line pierces a projecting plane, used in the two intermediate steps of Construction Two to obtain P_1 and P_2, should be used in all problems where applicable.

In orthographic projection the constructions for finding the point in which a line pierces a plane are utilized in obtaining the projections of any magnitude on an oblique plane (see Article 53). In architectural drawing they are used to cast shadows, and in pictorial drawing they are used to obtain oblique and perspective projections. Chapter VII deals with the applications of the principle to these last-mentioned fields of drawing.

53. To find the orthographic projection of a given line upon a given oblique plane when the line is not in the plane

Analysis. By definition, the orthographic projection of a line upon a given oblique plane is the line joining the piercing points, in the oblique plane, of the two perpendiculars dropped from the ends of the line to the plane. This line, in turn, must be represented by its projections on the coördinate planes.

Construction One. Given the line AB and the plane S in the first drawing of Fig. 69. From the point A, which is one end of the line, draw the line AC perpendicular to the plane S and of any length; $a'c'$ is perpendicular to Ss', and ac is perpendicular to Ss (Article 45). The line AC pierces the plane S at A_s, by Construction One of the preceding article. The projection of A_s upon the V plane is a_s'; upon the H plane it is a_s. The notation A_s is used instead of P for this piercing point because the point A_s is the projection of the point A on the plane S and should be lettered in conformity with the plan adopted for the projections on the coördinate planes. Similarly, draw the perpendicular BD and find the point B_s in which it pierces the plane S; the projections are b_s' and b_s. Join a_s' to b_s' and a_s to b_s. The line A_sB_s is the required projection of the line AB upon the plane S.

Construction Two. Given the line CD and the plane T in the second drawing of Fig. 69. Set up the auxiliary coördinate plane 3 perpendicular to the plane T (Article 47). Obtain c_3d_3 and t_3t_3 in the usual manner. From point C draw CE perpendicular to plane T; the projections are c_3e_3, $c'e'$, and ce. Produce c_3e_3 to intersect t_3t_3 at c_{t_3}, the auxiliary projection of the point C_t. By the proper return projecting lines, obtain c_t' and c_t, in order and on $c'e'$ and ce respectively. In the same way obtain d_{t_3}, d_t', and d_t by drawing the perpendicular DF with projections d_3f_3, $d'f'$, and df. Join C_t and D_t to obtain the required projection of the line CD on plane T.

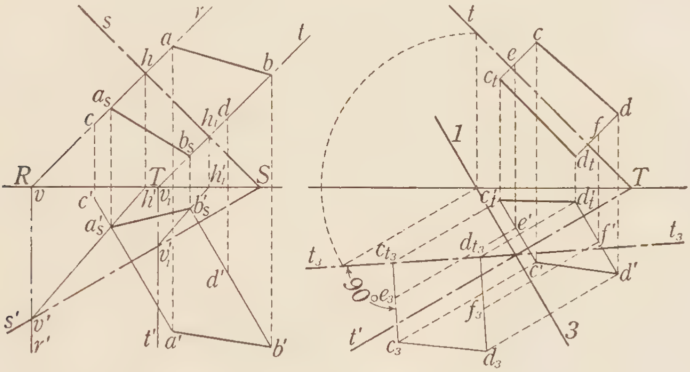


FIG. 69. Projection of a line upon a plane

54. Revolution of an oblique plane into a coördinate plane

It has already been explained how an oblique plane may be passed through any plane geometrical figure which is oblique to the coördinate planes and which is represented by two or more projections of its bounding lines. To carry out further constructions with this plane figure as a basis or to construct new figures in the same plane with it, it is necessary that the plane be revolved about one or another of its traces into the corresponding coördinate plane and all the drawings or geometrical figures on the plane be taken along with it and shown in their revolved positions. The new or revolved position of the plane shows the exact relations of all the lines in the plane to one another. The significant points in the plane are lettered, after their revolution, in correspondence with their designation in space, as other projections are, and the subscript letter r is attached to denote that the points are in revolved position and to keep them from becoming confused with the principal projections on the same sheet of paper. The turning movement just described is called the *revolution of an oblique plane.*

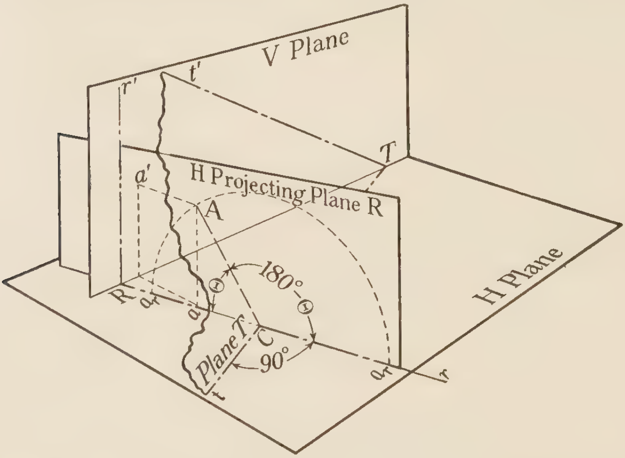


FIG. 70. Revolution of an oblique plane and a point in it

To illustrate the principle involved in any revolution of an oblique plane and to explain the constructions required, the plane T, in Fig. 70, has been revolved about its horizontal trace into the H plane, and a single point A in the plane has been shown in its alternate revolved positions as a_r, it being proper to turn the plane about its trace in either direction. The important things to be noted in the operation are

1. The trace Tt is the axis of revolution about which the plane T turns, either through the acute angle Θ or the obtuse angle $180° - \Theta$, at the will of the draftsman.

2. The point A in the plane T revolves in an H projecting plane R passed through the point A and perpendicular to the trace *Tt*.
3. The point A falls at a_r on a line in the H plane (H trace of R) through *a* and perpendicular to *Tt*, and at a distance from *Tt* equal to the radius of revolution, CA.
4. The radius of revolution, CA, is the hypotenuse of a right triangle, called the triangle of revolution, one side of which is the distance of A above the H plane (a' to the ground line), and the other side of which is the distance of the H projection, *a*, from *Tt*, the axis of revolution.

The space conditions portrayed in Fig. 70 have been put into the usual conventional scheme of representation in the first drawing of Fig. 71, where the plane T has been turned about its H trace into the H plane. In order that the triangle of revolution may be constructed conveniently and its hypotenuse used as a radius to obtain a_r, without any translation of distances to or from a separately constructed triangle, the plane R is first turned about its H trace into the H plane. The triangle of revolution is caa_x, corresponding to C*a*A in Fig. 70. A construction line has been substituted for the trace of the plane R in Fig. 71, merely as a

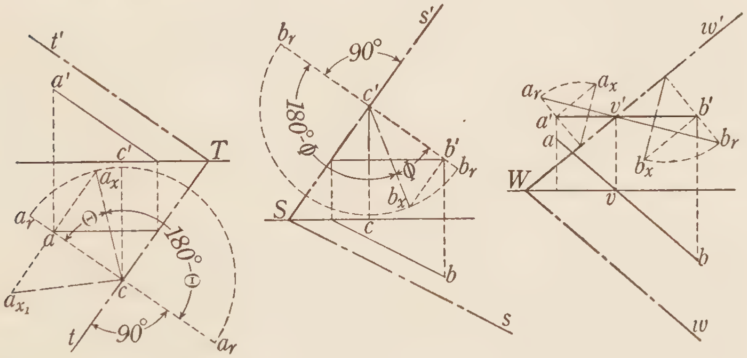


FIG. 71. Revolution of an oblique plane and a point in it

matter of maintaining uniformity. It should be noted from Fig. 70 that the plane R could have been turned into the H plane in the opposite direction, and the triangle of revolution would then have been as indicated by caa_{x_1} in the first drawing of Fig. 71.

In the second drawing of Fig. 71 the construction for revolving a plane about its vertical trace into the V plane has been shown. In this case the triangle of revolution is formed by using $b'c'$ as one side, $b'b_x$, equal to the distance of b from the ground line, as the other side, and the hypotenuse, $c'b_x$, to close the triangle and obtain the radius of revolution. The point b_x may be revolved in either direction to locate b_r, as in the first case.

In the third drawing of Fig. 71 the principle used in revolving a point has been applied to the line AB in the plane W. Here each end of the line has been revolved separately, beginning with the point B and using the V trace of the plane as the axis. Point B is in the plane W in front of the vertical plane, while point A is in the plane W and behind the vertical plane. Therefore point A must fall on the opposite side of the trace from point B in the revolved position. Point V, on the line AB, being in the vertical trace or axis, cannot move in the revolution of the plane; therefore a_rb_r must pass through v', thus forming a check on the accuracy of the constructions. Had the H trace been used as the axis, a_rb_r would have been parallel to Ww and passed through h, which is at infinity on Ww, again making a convenient, though partial, check on the construction work. It should be noted further that not only are the two triangles of revolution similar but their corresponding sides are parallel, thus checking the constructions in another manner. The line a_rb_r is the true length of the line AB.

It is seen that in the revolution of an oblique plane a method has been developed which will be found to have important applications in the matter of the construction of plane geometrical figures in any position in space. The only manipulations necessary are those of revolving the plane in which the figure is to be constructed into a coördinate plane, constructing the specified figure by the rules of plane geometry, then turning the plane with its constructed drawings back into its original position and showing the projections or views of this figure upon the coördinate plane. This last step is called *counter-revolution of an oblique plane.* Since the methods of counter-revolution are very important as working tools, several in number, and quite varied in character, they are presented in the form of Constructions under a general problem heading in the next article.

55. To construct any plane geometrical figure in a given oblique plane and about a point in the given plane as a center, and to show the projections of the figure

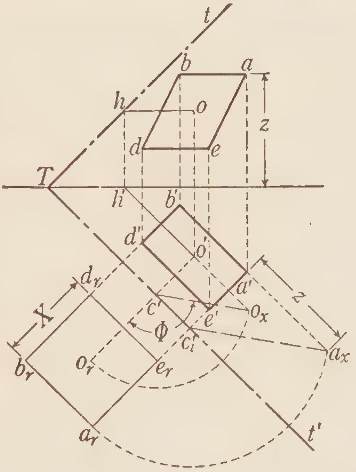


FIG. 72. Projections of a square in an oblique plane

Construction One. *By triangles of revolution.* Given the point O in the oblique plane T and the specification that a square, X inches on a side, is to be drawn in the plane with O as its center, Fig. 72. Revolve the plane T with the point O into the V plane about *Tt'* as an axis. The point O falls at o_r, by the procedure of Article 54. The square $a_rb_rd_re_r$, X inches on a side, is drawn about o_r as a center, as shown. Through a_r erect a perpendicular to *Tt'* and intersecting it at c_1'. With c_1' as a center and $c_1'a_r$ as a radius, describe the arc a_ra_x, equal in angular measurement to the arc o_ro_x. This will have been done if $c_1'a_x$ is drawn parallel to $c'o_x$, that is, when angle $a_rc_1'a_x$ becomes equal to angle $o_rc'o_x$. From a_x drop a perpendicular to a_rc_1' produced to intersect it at a', the vertical projection of the corner A of the required square. The H projection, a, is located above the ground line on a perpendicular through a', at a distance z, equal to $a'a_x$, from the ground line. In a similar manner locate the projections of the points B, D, and E. Join the corresponding projections in the proper order to obtain the complete projections of the required square.

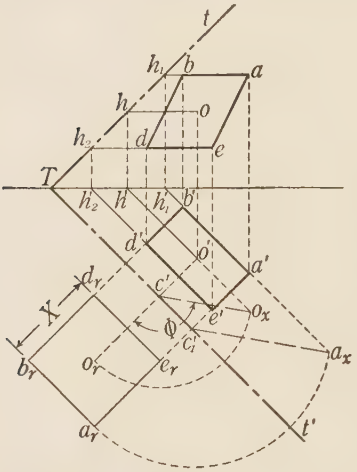


FIG. 73. Projections of a square in an oblique plane

Discussion. The horizontal projections of each point might have been readily determined, after each vertical projection was located, by the use of H- or V-parallels, as shown in Fig. 73, where V-parallels are used. It should be noted, further, that the angle $o_rc'o_x$ is the measure of the angle Φ that the plane T makes with the V plane and is, therefore, the angle all points in the plane T move through in the revolution and counter-revolution of the plane.

Construction Two. By general inclined lines. Given the point O in the oblique plane T and the specification that a square, X inches on a side, is to be drawn in the plane with O as its center, Fig. 74. Revolve the plane T with the point O into the H plane about *Tt* as an axis. Point O falls at o_r, by the procedure of Article 54. The square $a_rb_rd_re_r$, X inches on a side, is drawn about o_r as a center, as shown. Through o_r draw the diagonal a_rd_r, a definite line in the plane T. Extend a_rd_r to intersect the H trace of the plane at H. This point is the horizontal piercing point of the line

AOD and does not move in either the direct revolution or the counter-revolution. Its projections are *h* and *h'*. Draw the projections *oh* and *o'h'* of the line OH. Points A and D are on OH; hence their projections are on *oh* and *o'h'*. Through a_r and d_r extend perpendiculars to *Tt* until they intersect *oh* at *a* and *d* respectively. These are the horizontal projections of the two corners of the square. Erect vertical projecting lines through *a* and *d* to intersect *o'h'* at *a'* and *d'*, thus obtaining the vertical projections of the points A and D. In a similar manner draw the diagonal b_re_r and obtain the projections of the points B and E. Join both the H and V projections of the four points in turn by straight lines to obtain the projections of the required square.

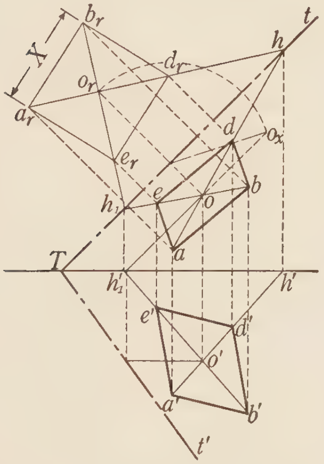


FIG. 74. Projections of a square in an oblique plane

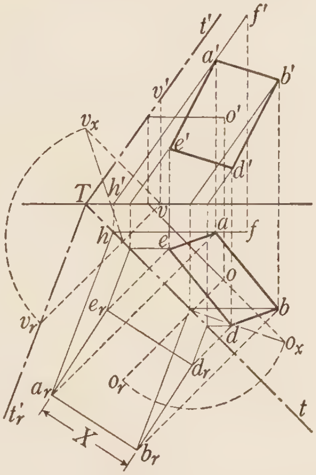


FIG. 75. Projections of a square in an oblique plane

Construction Three. By H- or V-parallels. Given the point O in the oblique plane T and the specification that a square, X inches on a side, is to be drawn in the plane with O as a center, Fig. 75. Revolve the plane T with the point O into the H plane about *Tt* as an axis and draw $a_rb_rd_re_r$ as in Constructions One and Two. Select any point V, with projections v' and v, in Tt', and revolve it about Tt into the H plane at v_r. Draw Tt_r' through v_r and the point T. This is the revolved position of the vertical trace. Any line in plane T parallel to Tt' is a V-parallel and falls parallel to Tt_r' in revolved position. The H piercing point of such a line, being in the axis of revolution, does not move in the direct revolution or the counter-revolution. Through a_r draw one such line and extend it to Tt to obtain the point H, with projections h and h'. Through h and h' draw the respective projections of this V-parallel, HF, and of any length. The point A is on the line HF, since hf_r, a portion of which is shown in the drawing, was drawn through a_r. Counter-revolve a_r by drawing through it a perpendicular to Tt and extending this line until it intersects hf at a, and by projecting vertically to $h'f'$ to obtain a'. The projections of the points B, D, and E are obtained by drawing other V-parallels and repeating the constructions used in obtaining the projections of the point A.

Construction Four. By H- or V-parallels. The squares, lettered $a_rb_rd_re_r$, of Figs. 72 to 75, inclusive, might be counter-revolved by still a fourth construction, not illustrated, in which H-parallels are used as the auxiliary lines with which to return individual points. For instance, in Fig. 75 a line parallel to Tt through a_r could be drawn and its v_r located in Tt_r' by extending it to that trace. This point could then be returned in the manner indicated for the single v_r of the figure. The projections of the H-parallel might then be drawn through the projections of the point V, and a_r returned to them as in Construction Three.

56. Oblique planes versus auxiliary coördinate planes in problem solutions

The method of obtaining the solutions of space problems by means of the oblique plane, with its attendant simple constructions, should be sharply differentiated from the more practical and more generally used method of obtaining them by means of auxiliary coördinate planes. Most problems may be solved by either method. In numerous cases one method has marked advantages over the other. In the remaining portions of this chapter and in other chapters, problems will be solved in the main by both methods, except where one or the other proves too cumbersome for practical use, or where other methods apply even better than either of them. They will be referred to as the *oblique-plane method* and the *auxiliary-plane method* respectively.

57. Auxiliary coördinate plane 4

Before it is possible to solve very advanced problems by the auxiliary-plane method, it is necessary to utilize a second auxiliary plane set up perpendicular to an auxiliary plane 3 which has already been established at right angles to an H, V, or P plane. Any auxiliary coördinate plane perpendicular to a plane 3 is designated as plane 4, and the projections of a point upon it are lettered the same as on other coördinate planes, with the addition of a subscript 4 to each letter to correspond to the number of the auxiliary coördinate plane. This second auxiliary plane is revolved about its intersection with the plane 3 as a ground line into coincidence with the 3-plane and then carried into the H, V, or P plane when plane 3 revolves about its own ground line. The new ground line is marked *3–4*, as shown in Fig. 76.

The projection a_4 in the first drawing of Fig. 76 is obtained by constructing a projecting line through a_3 perpendicular to *3–4* and laying off on it, from *3–4*, the distance of a from *0–3*. As stated in a previous article, this equality of distances between ground lines and corresponding projections depends upon the fact that if any two coördinate planes are set up perpendicular to a third coördinate plane, the projections of a point on the two planes will be found at equal distances from each of the two ground lines thus formed, after the planes have been revolved. In the first case of Fig. 76, the plane 4 and the H plane are both perpendicular to plane 3. Therefore the H projection of the point A is the same distance, n, from the ground line *0–3* that its projection on plane 4 is from the ground line *3–4*.

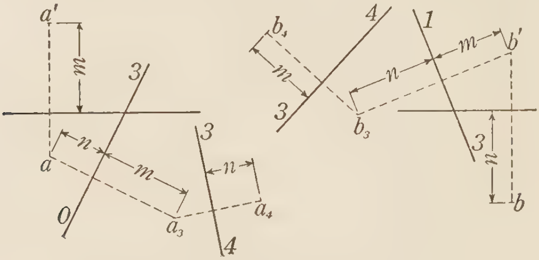


Fig. 76. Plane-4 projection of a point

In the second part of the figure, plane 4 and the V plane are each perpendicular to plane 3; therefore b' is the same distance from the ground line *1–3* that b_4 is from the ground line *3–4*.

58. Direction of measurement of auxiliary-plane projections

In obtaining the projections of one or more lines on either a plane 3 or a plane 4, care must be taken to determine on which side of the new ground lines the construction measurements are to be made. They will all be made on one side of a particular ground line only when the ends of the lines to be projected are all on one side of the coördinate plane to which the auxiliary plane is perpendicular. In Fig. 77, for instance, the line AB is partly above and partly below the H plane; hence a_3 falls on one side of *0–3*, the axis of revolution in the H plane, and b_3 on the other side. The H piercing point of AB projects into the line *0–3* and does not move in the revolution of the 3-plane, as shown by the position of h_3. This fact furnishes a check on the constructions for a_3 and b_3. The projection c_3 falls on the same side of *0–3* as does a_3, since the points A and C are both above the H plane. Again, in this same illustration the plane 3 has been set up in such a way that the line AC penetrates it at the point E, thus placing points A and B on the opposite side of the plane from C and causing c_4 to fall on the opposite side of *3–4* from a_4 and b_4. Here e_4, at the intersection of a_4c_4 with *3–4*, checks the constructions involved, as did h_3 in the case of the line AB.

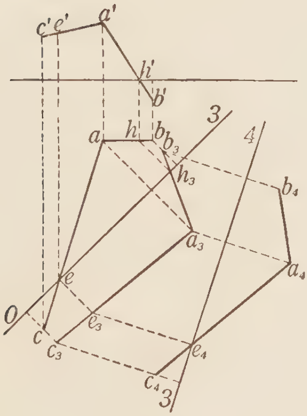


FIG. 77. Plane-4 projections of lines

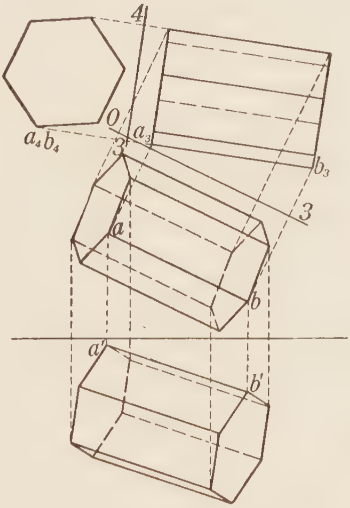


FIG. 78. Compact arrangement of a four-view drawing

No general inclusive rule can be laid down governing the location of planes 3 and 4 other than that the views should be kept from overlapping as much as possible. Fig. 78 shows a satisfactory arrangement of views in the case of the inclined hexagonal prism. In commercial drafting, ground lines and the lettering on views are almost entirely dispensed with. This improves the appearance of the drawing and focuses the attention on the shape and size of the object.

59. Direction of revolution of auxiliary coördinate planes

An auxiliary plane may be turned in either direction about its ground line. So long as only points and lines are involved, it will be a sufficient rule to follow that the plane be turned into the most available or open space on the drawing paper. In projecting solid objects or opaque surfaces, where the question of invisible lines is bound to arise, more restrictive rules must be adhered to. The drawings must be considered as views, that is, not merely abstract projections but things actually seen from some point of sight, and all lines on the object which are hidden from the observer must be represented by the usual broken-line or dash-line convention. This means that the point of sight, the location of the auxiliary planes, and their direction of revolution are fixed quite largely by practical considerations.

In the United States most commercial or technical drawings are made in the third quadrant; in England they are made in the first quadrant. Both first-quadrant and third-quadrant drawings are used in this text in connection with solid objects and opaque surfaces. The following definitions or rules apply with respect to the location of the point of sight and the direction of revolution of the auxiliary coordinate planes in all cases where practical considerations control. The profile plane must also fall under the scope of these rules, as was suggested in Article 7, because in effect it becomes an auxiliary coördinate plane when used in commercial drawing.

1. In the first quadrant, the object is always between the observer and the coördinate plane on which the object is being projected. In the third quadrant, the coördinate plane is always between the observer and the object.
2. In both first-quadrant and third-quadrant drawing, the direction of revolution of the coördinate plane is always away from the object.

3. The point of sight for each separate view (projection) is on a perpendicular to the coördinate plane used for obtaining the view and at an infinite distance from the plane, although for all practical purposes the observer may assume that he is only far enough away from the object to obtain its correct contour and edge lines. This means that the observer should imagine himself free to move from point to point in order that he may place his eye in line with the various perpendiculars to the coördinate planes.

It will be noted that rule 2 modifies the hitherto customary practice of extending the profile plane into all four quadrants and revolving it in one direction only. When the magnitude to be projected is a solid object or an opaque surface and is confined to a single quadrant, the profile or auxiliary planes are not extended into quadrants other than the one in which the object is located. Only in the more general derivations of theory and principle is it essential that these planes be considered unlimited in extent or fixed in direction of revolution.

60. Placing of auxiliary coördinate planes 3 and 4

In the previous discussions of auxiliary coördinate planes, little has been said concerning the methods of setting up these planes in any but perfectly general positions. In Article 19 it was shown that a plane 3 may be placed parallel to a given line by drawing its ground line parallel to the corresponding projection of the line. In a similar manner plane 4 may be set up parallel to any line. This scheme was used, in the case of the plane 3, to find the true length of a line.

In Article 47 an auxiliary plane was set up perpendicular to an oblique plane to obtain an edgewise view of the plane or of any geometrical figures in it. The most important practical uses of the auxiliary plane, however, are to be found in its applications to problems which require it to be placed perpendicular to a line or parallel to the plane of two or more lines. The first application gives the so-called point-projection of a line, while the second gives the true angle between lines, areas of plane figures, and other necessary information in technical drawing. The methods and devices most commonly used to set up auxiliary planes in the positions indicated above are presented in the subsequent demonstrations.

61. The auxiliary coördinate plane perpendicular to a line

In the first drawing of Fig. 79, a 3-plane has been established perpendicular to the line AB by inspection, since the line AB is a plane-parallel (Article 50), and a_3b_3 has been obtained as the point-projection of the line on plane 3. In the second drawing of the figure, the line CD is inclined to all three primary planes; hence plane 3 is first set up parallel to CD and the projection c_3d_3 is obtained in the usual way. Plane 4 may now be set up perpendicular to the line, since the line is parallel to plane 3, by drawing the ground line, *3–4*, perpendicular to c_3d_3. The point-projection of the line CD on plane 4 is c_4d_4. The relation between c_3d_3 and c_4d_4 is the same as that of the vertical and horizontal projections of a line perpendicular to either the H or the V plane.

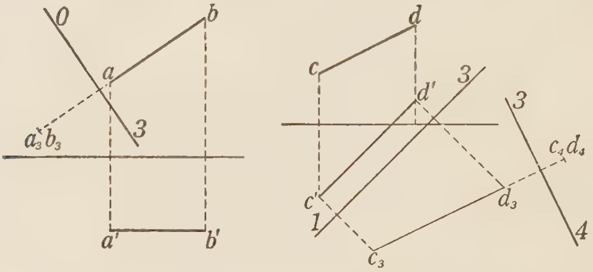


FIG. 79. Auxiliary plane perpendicular to a line

62. The auxiliary plane parallel to the plane of two or more lines

In Fig. 80 the lines AB and BC are any two intersecting lines in space. The line CD is drawn parallel to the H plane and connecting the two lines. Plane 3 is set up perpendicular to CD as in Fig. 79, thus becoming perpendicular to the plane of AB and BC by the principle of solid geometry that a plane is perpendicular to another plane if it is perpendicular to any line in the second plane. Hence AB and BC will project on plane 3 as a straight line, $a_3b_3c_3$, as shown.

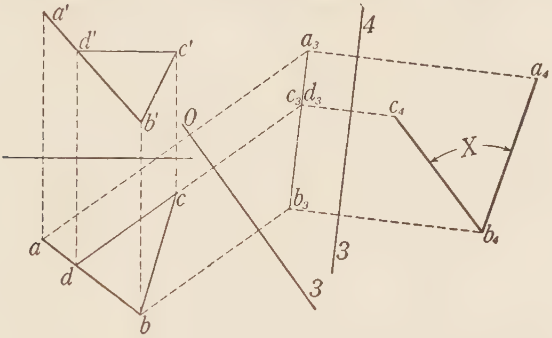


FIG. 80. Auxiliary plane parallel to plane of two lines

Plane 4 can now be set up parallel to both AB and BC by drawing its ground line, *3–4*, parallel to $a_3b_3c_3$. Plane 4, therefore, is parallel to the plane of AB and BC, and $a_4b_4c_4$ shows the true angle X between the lines, as well as the true length of each line.

63. The true length of a line

The use of the auxiliary plane for finding the true length of a line was demonstrated in Article 19. In revolving the oblique plane of Article 54, a second method was demonstrated, because all lines in a plane are shown in their true lengths when the plane is revolved into one of the coördinate planes. Still a third method should be demonstrated, since it is practical and requires little construction detail.

Let the line AB, in the first drawing of Fig. 81, be revolved about either the H or V projector of one extremity until it is parallel to the V or H coördinate plane respectively. Its projection on that plane to which it becomes parallel will show its true length. In the figure the H projector of the point A is used as the axis and the line is turned about it, with point A fixed, until the line is parallel to the V plane. The point B does not rise or fall in the turning movement; hence its V projection moves only to the right or left horizontally, while its H projection moves in the arc of a circle, with ab as a radius, until ab_r becomes parallel to the ground line. The projections of the line AB in its new position are ab_r and $a'b_r'$. The true length of the line is $a'b_r'$. The letters *T. L.* (true length) or *T. D.* (true distance) are used to identify this true-length projection on a drawing.

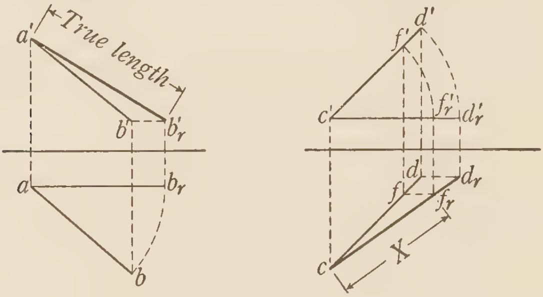


FIG. 81. True length of a line

The second part of Fig. 81 shows the method of construction when a V projector of one end of the line is used as the axis of revolution, and also it shows the method of laying off a specified distance on a line of any length. A distance of X inches has been measured from the point c to f_r on the true-length projection, cd_r, and the projections of the point F, thus established, have been found by reversing the revolution process used in obtaining the true length of the line CD.

64. To find the shortest distance from a given point to a given plane

Analysis. The shortest distance from a given point to a given plane is the perpendicular distance from the given point to the given plane, or, in other words, it is the true length of this perpendicular from its piercing point in the plane to the given point.

Construction One. Given the point O and the plane T in the first drawing of Fig. 82. Draw the line ON perpendicular to the plane T. The line ON pierces the plane T at the point P, by Construction One of Article 52. The true length of the line OP is $o'p_r'$, by the construction of the preceding article, which, therefore, is the shortest distance from the point O to the plane T.

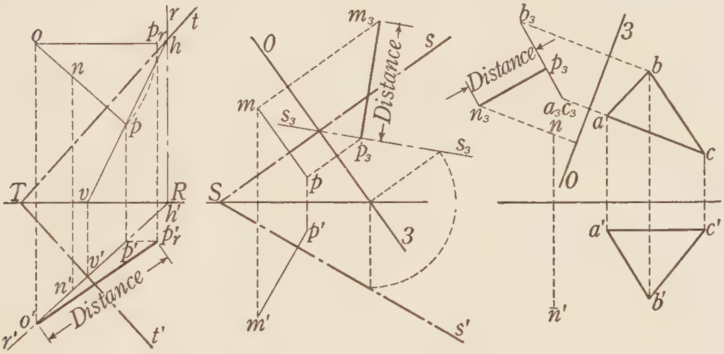


FIG. 82. Distance from a point to a plane

Construction Two. Given the point M and the plane S in the second drawing of Fig. 82. Set up the auxiliary coördinate plane 3 perpendicular to the plane S. The projection of the point M on plane 3 is m_3, and s_3s_3 is the trace of plane S on the auxiliary plane, by the usual constructions. Since the edgewise view of plane S on plane 3 is s_3s_3, and since the perpendicular from the point M to the plane S is parallel to plane 3, by construction, the perpendicular distance from the point M to the plane S shows directly on plane 3 as

m_3p_3. The projections of MP are $m'p'$ and mp, each being drawn perpendicular to the corresponding trace of plane S.

Construction Three. Given the point N and the plane triangle ABC in the third drawing of Fig. 82. Set up the auxiliary plane 3 perpendicular to the plane of the triangle by making *0–3* perpendicular to *ac* (see Article 62). The projections of the point N and the triangle ABC on plane 3 are n_3 and $a_3b_3c_3$ respectively. Since the perpendicular from the point N to the plane of the triangle is parallel to plane 3, by construction, and since $a_3b_3c_3$ is the edgewise view of the given triangle, the perpendicular distance from the point to the triangle shows directly on plane 3 as n_3p_3.

65. To find the shortest distance between two parallel planes

Analysis. The shortest distance between two parallel planes is the true length of a line drawn perpendicular to the planes, beginning at a chosen point in the one plane and ending at the piercing point of the line in the other plane.

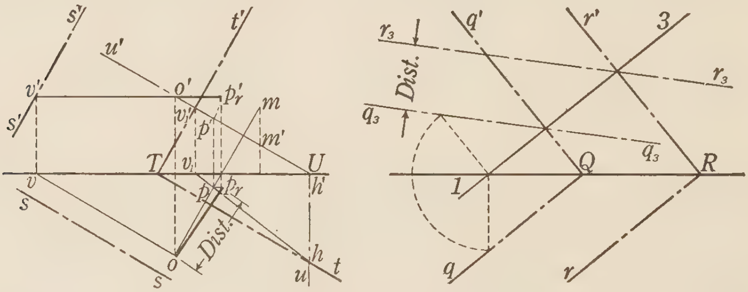


FIG. 83. Distance between two parallel planes

Construction One. Given the two parallel planes S and T in the first drawing of Fig. 83. Select the point O in the plane S and draw the line OM perpendicular to the two planes. The line OM pierces the plane T at the point P. The true length of OP is op_r, the required distance between planes S and T.

Construction Two. Given the two parallel planes Q and R in the second drawing of Fig. 83. Set up the auxiliary coördinate plane 3 perpendicular to the two planes. The auxiliary traces r_3r_3 and q_3q_3 are the edgewise views of the planes on plane 3; therefore any perpendicular between the planes will show in true length as a perpendicular line between the traces r_3r_3 and q_3q_3.

66. To find the projections and true length of the shortest line from a given point to a given line

Analysis. The shortest distance from a point to a line is the perpendicular distance from the point to the line. This distance will show truly on an oblique plane passed through the point and line after the plane has been revolved into a coördinate plane, or it will show directly on an auxiliary coördinate plane set up perpendicular to the given line as the distance between the point-projection of the line and the projection of the given point on the auxiliary plane. The projections of the perpendicular from the point to the given line will not be perpendicular to the corresponding projections of the line (see Article 46).

NOTE. If the given line is parallel to a coördinate plane, the perpendicular may be drawn directly, as explained in Article 20, and the true length found by any of the methods referred to in Article 63.

Construction One. Oblique-plane method. Given the point O and the line AB in the first drawing of Fig. 84. Pass the plane T through the point O and the line AB by joining the point O to the point B and then passing the plane through the lines AB and BO. Only one trace need be drawn as an axis. Revolve the plane T, with the point O and the line AB, about Tt' into the V plane. The point B falls at b_r, O falls at o_r on $v_1'b_r$ produced, and A falls at a_r on $v'b_r$. Draw o_rm_r perpendicular to a_rb_r. It is the true length of the required perpendicular. The point M is on AB, since m_r is on a_rb_r; therefore, when M counter-revolves into its space position, its projections, m and m', will be on ab and $a'b'$ respectively. The line OM is the required perpendicular.

Construction Two. Auxiliary-plane method. Given the point K and the line CD in the second drawing of Fig. 84. Project the point and line upon the auxiliary coördinate plane 3 which has been set up parallel to the line CD. These projections are k_3 and c_3d_3 respectively. Set up auxiliary plane 4 perpendicular to the line CD; the ground line *3–4* must be perpendicular to c_3d_3, by Article 61. The projections of the point K and the line CD on plane 4 are k_4 and c_4d_4 respectively. The projection k_4n_4 shows the true length of the required perpendicular from K to CD. Since CD is parallel to plane 3, k_3n_3 must be perpendicular to c_3d_3, by Article 20. The projections n and n' are returned from n_3 in the usual manner and the H and V projections of KN are drawn to complete the construction.

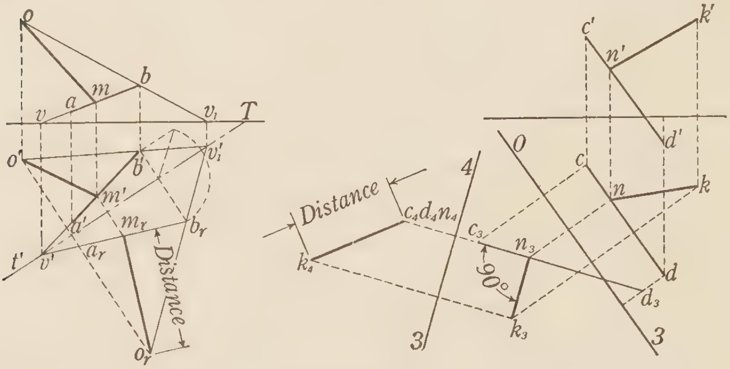


FIG. 84. Distance from a point to a line

67. To find the shortest distance between two parallel lines

Analysis. The shortest distance between two parallel lines is the perpendicular distance between them. This distance will show truly on an oblique plane containing the two lines after the plane has been revolved into a coördinate plane, or it will show directly as the distance between the point-projections of the two lines on an auxiliary coördinate plane that is set up perpendicular to the lines.

Construction One. *Oblique-plane method.* Given the two parallel lines AB and CD in the first drawing of Fig. 85. Pass the plane T through the two lines AB and CD. Only one trace need be drawn. Revolve the plane T with the two lines about $t't'$ into the V plane. The line AB falls at a_rb_r; the line CD falls at c_rd_r. The perpendicular distance between a_rb_r and c_rd_r is the required distance between the two given lines.

Construction Two. *Auxiliary-plane method.* Given the two parallel lines EF and GK in the second drawing of Fig. 85. Set up the auxiliary coördinate plane 3 parallel to the two given lines and project them upon it at e_3f_3 and g_3k_3. Set up the auxiliary plane 4 perpendicular to the two given lines and project them upon it at e_4f_4 and g_4k_4. The distance between e_4f_4 and g_4k_4 is the required distance.

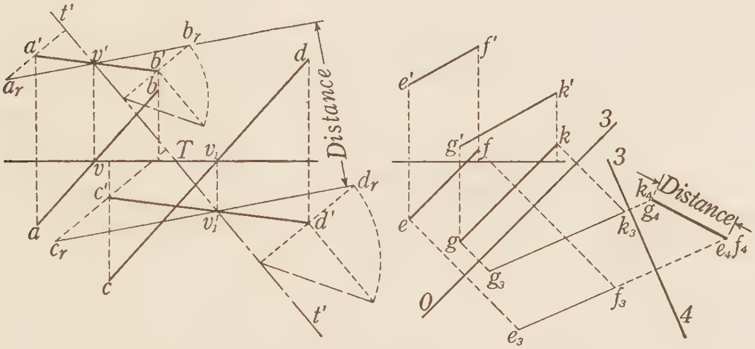


FIG. 85. Distance between two parallel lines

68. To find the true size of the angle X between two intersecting lines

Analysis. The true value of the angle X between two given intersecting lines will show on an oblique plane containing the two lines when the oblique plane is revolved into a coördinate plane, or it will show directly between the projections of the two lines upon an auxiliary coördinate plane set up parallel to the plane of the two intersecting lines.

Construction One. Oblique-plane method. Given the two intersecting lines AB and BC in the first drawing of Fig. 86. Pass the plane T through the two lines. Only one trace need be drawn. Revolve the plane T about *tt* into the H plane. The point B, the vertex of the angle X, falls at b_r. Join b_r to h and h_1, the points on the two lines which do not move in the revolution of plane T. The angle hb_rh_1 is the required angle X.

Construction Two. Auxiliary-plane method. Given the intersecting lines DE and EF in the second drawing of Fig. 86. Project the two lines upon the auxiliary coördinate plane 3 which is set up perpendicular to the plane of the lines. This is done by drawing the H-parallel, FC, intersecting the two given lines and then drawing the ground line *0–3* perpendicular to *fc*. The given lines project as a straight line, $d_3e_3f_3$, on this plane. Set up plane 4 parallel to the two given lines by making the ground line *3–4* parallel to $d_3e_3f_3$ and project the lines upon this plane at $d_4e_4f_4$. Angle $d_4e_4f_4$ is the required angle X.

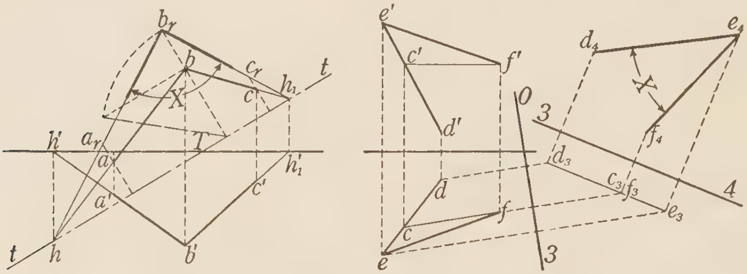


FIG. 86. Angle between two intersecting lines

69. To find the projections of the bisector of the angle between two given lines

Analysis. The bisector of an angle lies in the plane of the two lines forming the angle. Its projections do not bisect the angles formed by the projections of the lines. The bisector will show in true position on an oblique plane containing the two lines when the oblique plane is revolved into a coördinate plane, or it will show directly on an auxiliary coördinate plane set up parallel to the plane of the two intersecting lines. Its projections may be obtained by counter-revolution of the oblique plane or by the return of the auxiliary projection to H and V projections.

NOTE. The bisector of the angle between two intersecting lines is the locus of points that are equidistant from the two lines and in their plane.

Construction One. *Oblique-plane method.* Given the two intersecting lines AB and BC in the first drawing of Fig. 87.

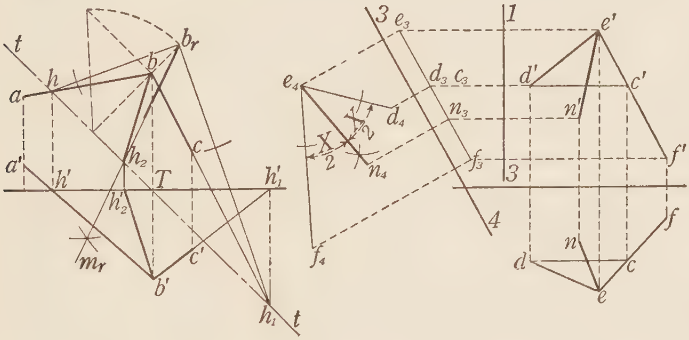


FIG. 87. Bisector of angle between two intersecting lines

Pass the plane T through the two lines. Only one trace need be drawn. Revolve the plane T about Tt into the H plane. The point B, the vertex of the angle between the two lines, falls at b_r. Join b_r to h and h_1, the H projections of the H piercing points of the lines AB and BC respectively. The angle hb_rh_1 is the true angle between the given lines. Draw the bisector, b_rm_r, of the angle hb_rh_1. Counter-revolve b_rm_r by using the H piercing point, H_2, of the line BM. Draw bh_2 and $b'h_2'$. The line BH_2 is the required bisector.

Construction Two. *Auxiliary-plane method.* Given the two intersecting lines DE and EF in the second drawing of

Fig. 87. Project the two lines upon the auxiliary coördinate plane 3 set up perpendicular to the plane of the two lines; then project them upon a plane 4 set up parallel to the plane of the two lines to obtain the angle $d_4e_4f_4$ which the lines make with each other. Draw the bisector, e_4n_4, of the angle $d_4e_4f_4$. The point N returns from its plane-4 projection, n_4, to its plane-3 projection at n_3, and to its V and H projections at n' and n respectively. The line EN is the required bisector.

70. To draw a line through a given external point and making a specified angle X with a given line

Analysis. Two lines may be drawn which will satisfy the given conditions, unless the given angle is 0° or 90°; then only one line may be drawn. The required lines will lie in the plane of the given point and line. The specified angle between the given line and the line to be drawn will show in its true value on this oblique plane when it has been revolved into a coördinate plane, or the angle will show directly on an auxiliary coördinate plane parallel to the plane of the given point and line. The projections of the required line may be found by counter-revolution of the oblique plane or by the return of the auxiliary projection to H and V projections.

Construction One. Oblique-plane method. Given the line AB and the point C in the first drawing of Fig. 88. Pass the oblique plane T through the line AB and the point C by joining the points C and K to form two intersecting lines. Revolve the plane T into the H plane about tt. The line AB falls along hk_r, although only a_r is shown revolved in the figure. The point C falls at c_r on k_rh_1. Draw c_rm_r making the required angle X with k_rh. The point M on the line KH — m_r is on k_rh — counter-revolves into its space position with H and V projections at m and m' respectively. Join the point C to the point M. The line CM is the required line. A second line, CN, satisfying the given requirements can be obtained by drawing c_rn_r to hk_r extended and making the angle X with it.

Construction Two. Auxiliary-plane method. Given the line CD and the point E in the second drawing of Fig. 88. Project the line CD and the point E upon the auxiliary coördinate plane 3, which is set up perpendicular to the plane of CD and E. This is done by drawing *0–3* perpendicular to *ef*, the H projection of an H-parallel joining the point E to the line CD. Then project the line and the point upon plane 4, which is set up parallel to the plane of the given point and line by drawing *3–4* parallel to $c_3e_3d_3$. Draw e_4n_4 making the required angle with c_4d_4. The projection e_4n_4 returns in the usual manner to give the H and V projections of EN, the required line.

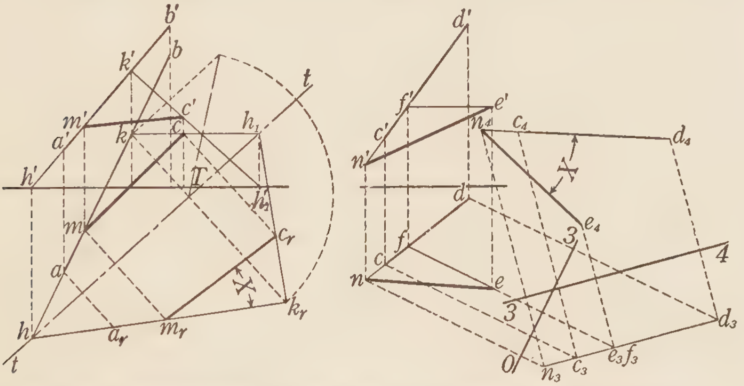


FIG. 88. Line through a point making a specified angle with a line

71. To find the acute angle X between a given line and plane

Analysis. The angle between a given line and plane is the angle between the line and its projection on the given plane. The given line extended until it pierces the plane, its projection on the given plane, and the perpendicular dropped from any point on the line to the plane, form a right triangle, one acute angle of which is the required angle and the other acute angle of which is the complement of the required angle. This latter angle is the more readily constructed.

Construction One. *Oblique-plane method.* Given the line AB and the plane S in the first drawing of Fig. 89. Draw the line BC perpendicular to the plane S. The angle ABC is the complement of the required angle between the line AB and the plane S. Pass the plane R through the lines AB and BC and revolve the plane about *rr* into the H plane.

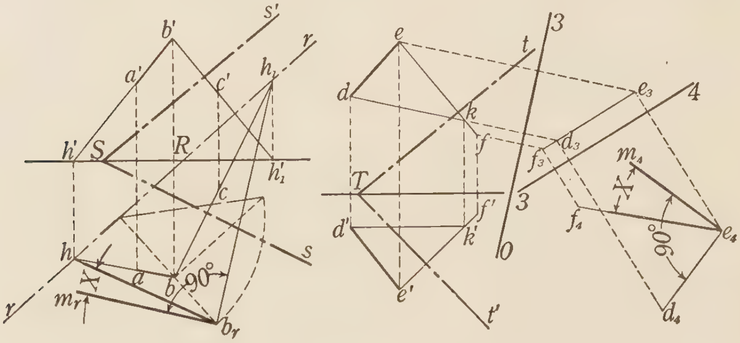


FIG. 89. Angle between a line and a plane

Point B, the vertex of the angle to be measured, falls at b_r. Join b_r to h and h_1, the respective H projections of the H piercing points of the lines AB and BC, which points do not move in the revolution. Angle hb_rm_r is the required angle X, since it is the complement of the angle hb_rh_1, the true value of the angle ABC, which would be identical with the angle $a_rb_rc_r$ were points A and C used in the revolution.

Construction Two. *Auxiliary-plane method.* Given the line DE and the plane T in the second drawing of Fig. 89. Draw the line EF perpendicular to the plane T. The angle DEF is the complement of the required angle between the line DE and the plane T. Set up the auxiliary coördinate plane 3 perpendicular to the plane of the lines DE and EF.

This is done by drawing *0–3* perpendicular to *dk*, the H projection of an H-parallel joining the lines DE and EF. The projections of the two lines on plane 3 are d_3e_3 and e_3f_3. Set up auxiliary plane 4 parallel to the plane of the lines DE and EF. This is done by drawing *3–4* parallel to $f_3d_3e_3$. The projections of DE and EF on this plane are d_4e_4 and e_4f_4. Angle $f_4e_4m_4$ is the required angle X, since it is the complement of $f_4e_4d_4$.

Discussion. If the line BC in Construction One or the line EF in Construction Two had been drawn in the opposite direction, an obtuse angle would have been obtained in each construction as the complement of the required angle. Since the angle desired is the acute angle between the given line and plane, it may readily be obtained by subtracting 90° from the obtuse angle.

Since the projection of a line on a given plane lies in the line of intersection of the given plane with a plane that is passed through the line perpendicular to the given plane, it is often advantageous, in using Construction One, to measure the angle between the given line and this line of intersection directly to obtain the angle X.

72. To find the angle X between two given planes

Analysis. By definition, the angle between two given planes is the angle between two lines which have been cut from the planes by a third plane passed perpendicular to the line of intersection of the two given planes. Or if, from some chosen point in space, a perpendicular is drawn to each plane, these two perpendiculars and the two lines cut from the given planes by the plane of the perpendiculars form a plane quadrilateral with two of its angles right angles and the other two angles supplements of each other. Since each of these two supplementary angles of the quadrilateral is equal to one or the other of the two angles formed by the given planes when extended indefinitely, either angle may be measured and the other derived by subtraction from 180°.

NOTE. In the use of trigonometric tables it is always desirable to have the angles expressed as acute angles; hence the smaller angle between two planes should be chosen if the measurement of the angle is to be expressed in degrees. In practical work the obtuse angle between planes is as likely to be needed as the acute angle between them.

Construction One. Oblique-plane method. Given the planes R and S in the first drawing of Fig. 90. The line of intersection of the two planes is VH. Through O, any chosen point on VH, pass the plane T perpendicular to VH, by means of the H-parallel OV_1. Only one trace of T is needed. Plane T intersects plane R in the line OV_2; plane T intersects plane S in the line OV_3. These lines are shown in the figure only in revolved position as o_rv_2' and o_rv_3', respectively, to avoid confusion in construction. The angle V_2OV_3 is one of the supplementary angles between the given planes. Its true size shows in the drawing as $v_2'o_rv_3'$, marked X_1, and either this value or its supplement, marked X, is the required angle.

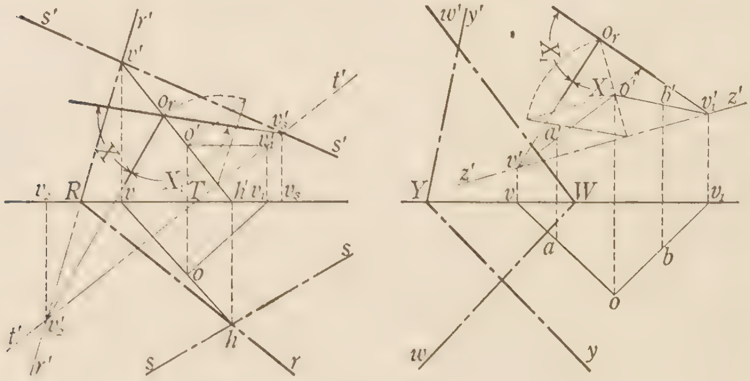


FIG. 90. Angle between two planes

Construction Two. Oblique-plane method. Given the planes W and Y in the second drawing of Fig. 90. The point

O is chosen at any convenient location in space. Draw the line OA perpendicular to the plane W and OB perpendicular to the plane Y. Pass the plane Z through the lines OA and OB and revolve it, with the two lines, about $z'z'$ into the V plane. The angle $v'o_rv_1'$ is the required angle X.

Construction Three. *Auxiliary-plane method.* The only practical application of the auxiliary-plane method to this problem, when the given planes are represented by traces, is in measuring the angles between the two intersecting lines of Construction One or Construction Two. This procedure is fully demonstrated in Article 68 and in subsequent articles; hence it will not be repeated as a separate construction here. When the given planes are represented as definite limited surfaces, as in Fig. 91, the auxiliary-plane method is directly applicable in obtaining the angles between the surfaces.

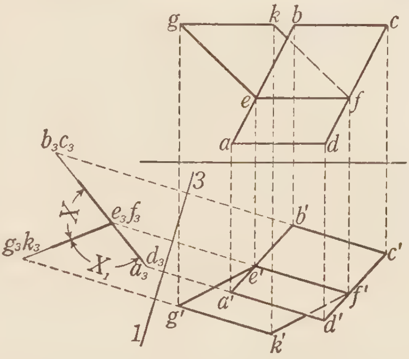


FIG. 91. Angle between two plane surfaces

The two parallelograms of Fig. 91 intersect in the line EF. If an auxiliary coördinate plane is set up perpendicular to the line EF, the two surfaces will project upon it as two straight lines; that is, the edgewise views of the two surfaces will be obtained. Since the line of intersection, EF, is a V-parallel, coördinate plane 3, with the ground line *1–3* perpendicular to $e'f'$, may be set up perpendicular to EF directly, without the intermediate step of first projecting the parallelograms upon an auxiliary coördinate plane parallel to EF. The projections of the two parallelograms, ABCD and EFKG, on plane 3 fall in the lines b_3a_3 and g_3e_3 respectively. Angle X is the required angle.

Discussion. Had the line of intersection EF of the two surfaces of the preceding Construction been unknown, it would have been necessary to find it before any auxiliary coördinate plane could have been set up. In most instances in commercial drafting the line of intersection will be known. If it should not be an H- or a V-parallel, then an auxiliary coördinate plane 4 will be necessary before the edgewise views of the surfaces can be obtained.

73. To find the plane-bisector of either dihedral angle between two given planes

Analysis. The plane which bisects either dihedral angle between two given planes contains the line of intersection of the two planes. It also contains a line which bisects the angle between the two lines that are cut from the given planes by a third plane passed perpendicular to the line of intersection of the two given planes (Article 72). There are two possible solutions, and since the angles between the two planes are supplementary, the required plane in the second solution must be perpendicular to the required plane of the first solution and hence perpendicular to the bisecting line.

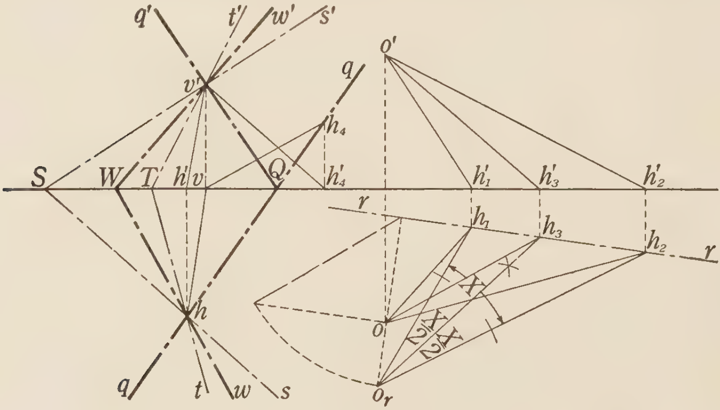


FIG. 92. Plane-bisectors of angles between two planes

Construction. Given the planes S and T, Fig. 92. The line of intersection of the two planes is VH. From the chosen point O in space draw the lines OH_1 and OH_2 perpendicular to the planes S and T respectively. Pass the plane R through the lines OH_1 and OH_2 and revolve it about rr into the H plane to obtain the angle $h_1o_rh_2$, the angle between the two lines and hence one of the angles between the planes S and T. Bisect the angle $h_1o_rh_2$ by drawing o_rh_3. The projections of the bisector OH_3 are oh_3 and $o'h_3'$, as shown in the figure. The line OH_3 is parallel to any line-bisector of one of the angles between the planes S and T. Draw the line VH_4 parallel to OH_3. Plane Q, passed through the lines VH and VH_4, is one of the required planes. Plane W, passed through the line VH and perpendicular to VH_4, is the second required plane.

NOTE. The plane-bisector of the dihedral angle between two planes is the locus of points equidistant from the two planes.

74. To pass a plane through a given line in a given plane so that the two planes will make a specified angle X with each other

Analysis. The required plane will contain the given line and a second line intersecting it that makes the specified angle with the given plane. This latter line lies in a plane perpendicular to the given line and is so drawn that it makes the required angle with the line cut from the given plane by the perpendicular plane.

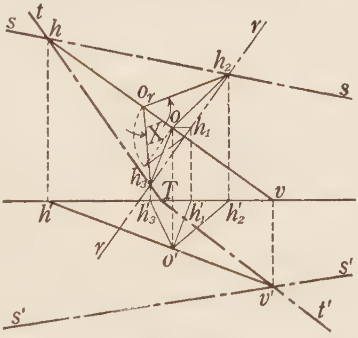


FIG. 93. Plane through a line, making a specified angle with a second plane

Construction. Given the line HV in the plane S, Fig. 93. Through any chosen point O on the line HV pass the plane R perpendicular to HV, by means of a V-parallel through O. Only the H trace, *rr*, need be drawn. The plane R cuts the line OH_2 from the plane S. Revolve the plane R with the line OH_2 about *rr* into the H plane, giving o_rh_2 as the revolved position of the line. Draw o_rh_3 making the specified angle X with o_rh_2. The plane T, containing the lines VH and OH_3, is the required plane.

75. The angles Θ, Φ, and Π of lines and planes

It is so often necessary to determine the angle a given line or plane makes with one or another of the coördinate planes, or, conversely, to draw a line or plane that makes certain given angles with the coördinate planes, that special symbols are chosen to designate these angles. They are the Greek letters Θ, Φ, and Π.

The angle between any line or plane and the H plane is designated by the symbol Θ; the angle between a line or plane and the V plane is designated by the symbol Φ; the angle between a line or plane and the P plane, by the symbol Π. The horizontal and vertical bars of Θ and Φ are readily associated with the corresponding plane.

In Article 71 the angle between a line and a plane was defined as the angle between the line and its orthographic projection on the plane. Hence, in determining the angles Θ, Φ, and Π of a given line, it is necessary to measure the angle between the given line and its projection on one or another of the coördinate planes. It was shown in Article 71, in measuring the angle between a given line and an oblique plane, that it saves time to measure the angle between the given line and a perpendicular from some point on it to the given plane and then to obtain the complement of this angle as the required angle. In the case of an angle between a coördinate plane and a given line, it is better to proceed by the direct method, as suggested in the definition.

76. To find the angles Θ, Φ, and Π of a given line

Analysis. By definition, the angles between the given line and its three primary projections must be found. A plane passed through a line and its projection on any coördinate plane is always perpendicular to that coördinate plane; in other words, it is a projecting plane. If each projecting plane of a line is revolved about one or another of its traces into the corresponding coördinate plane, carrying with it both the line and the projection which are to determine a particular angle, the true size of each required angle will show on the coördinate plane as the angle between the revolved position of the line and the corresponding revolved position of its projection.

Construction One. Given the lines AB, CD, and EF, Fig. 94. Pass the H projecting plane R through the line AB and its H projection, *ab*; pass the V projecting plane S through the line CD and its V projection, $c'd'$; pass the P projecting plane T through the line EF and its P projection, e_2f_2. Revolve the line and the projection named in each case about the inclined trace of the projecting plane into the corresponding coördinate plane. The projection involved in each instance will not move, since it is in the trace or axis of revolution. The given lines revolve to the positions indicated in the figure, and the angles between the revolved positions and the traces that were used as axes are the required angles Θ, Φ, and Π, as shown.

It should be observed that a check on each construction is obtained by means of one of the piercing points of the line, which does not move in the revolution, and which might have been used instead of one of the given points on the line in obtaining the revolved position.

Discussion. In practical work the traces of the projecting planes are omitted altogether and the projections of the line are used as the axes of revolution, as shown for the lines GK, MN, and OQ in the last three drawings of Fig. 94.

Sometimes the line is projected upon auxiliary coördinate planes set up parallel to, or through, the line, and the required angles are then measured between the new projections and the plane-3 ground lines, instead of between projections and revolved positions, as is done in the examples of Fig. 94. The results, of course, are identical.

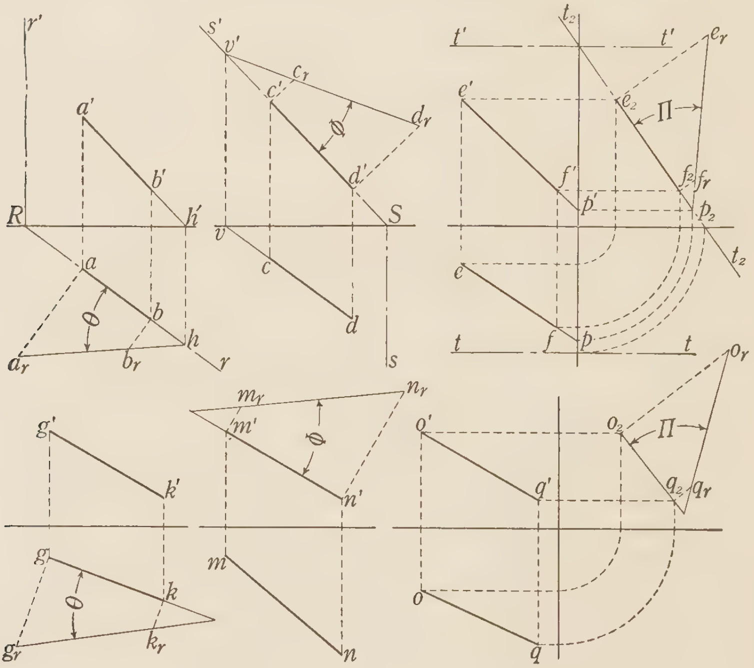


FIG. 94. Angles Θ, Φ, and Π of a line

Construction Two. If, in Fig. 94, the perpendicular trace had been used as the axis of revolution in each case, the constructions would have been as shown in Fig. 95. Here a short-cut method is employed to obtain the revolved position of the line and its projections. This may be understood by observing that the triangle of revolution for each point is duplicated by one already drawn which is bounded by the ground line, the inclined trace of the plane, and a projecting line, as shown for the point B in the first drawing of the figure. The angle between $a_r'b_r'$, the revolved position of the line, and a_rb_r, the revolved position of the H projection of the line, in the first drawing of the figure, is the required angle Θ. The angles Φ and Π are obtained similarly, as shown in the second and third drawings of the figure.

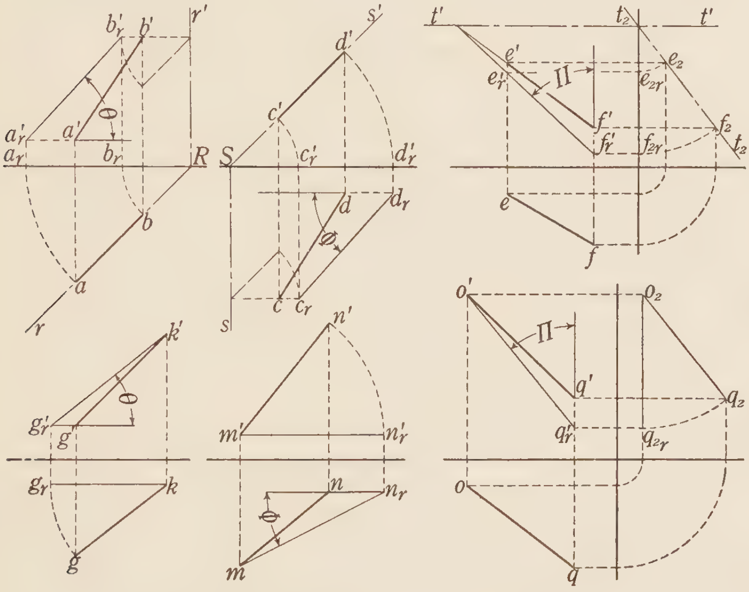


Fig. 95. Angles Θ, Φ, and Π of a line

Discussion. As in Construction One, a practical adaptation of the construction is used in commercial work to shorten the labor involved. This is shown in the last three drawings of the figure.

The methods of Constructions One and Two may be used in a converse way to draw lines in space which make given angles with one or another of the coördinate planes. The student should make the adaptations in order that he may compare them with the so-called cone method of drawing such lines, which is explained in Article 111.

77. To find the projections and true length of the common perpendicular between two non-parallel, non-intersecting lines

Analysis. Only one perpendicular may be drawn between two non-parallel, non-intersecting lines. It is perpendicular to a plane passed through one of the given lines, parallel to the second line. It originates at the point of intersection of the first line with the projection of the second line upon the plane. The perpendicular to the plane through this initial point on the first given line intersects the second line and is perpendicular to them both. The true length of the constructed perpendicular may then be found. Such a line does not have its projections perpendicular to the respective projections of the two given lines (see Article 46).

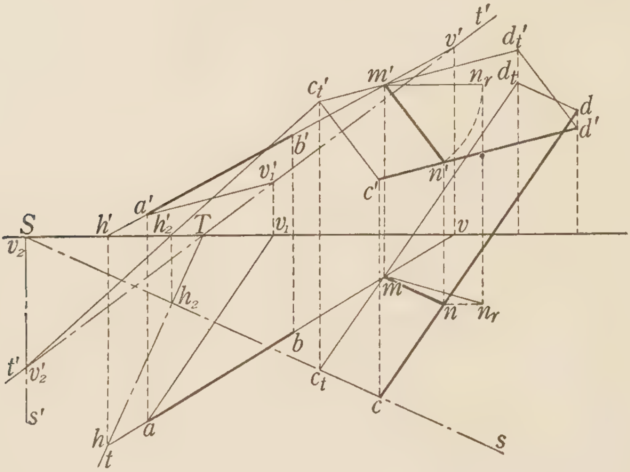


FIG. 96. Distance between two non-parallel, non-intersecting lines

The perpendicular between the two given lines also shows in true length, and as a point-to-projection perpendicular, on an auxiliary coördinate plane which is set up perpendicular to one of the lines, because such an auxiliary plane is parallel to the common perpendicular between the two lines.

Construction One. Given the two lines AB and CD, Fig. 96. Pass the plane T through the line AB and parallel to the line CD, by Article 42. Project the line CD upon the plane T at C_tD_t, by Article 53. Only the construction for obtaining C_t need be carried out, since the line C_tD_t must be parallel to CD, and its projections must, therefore, be drawn parallel to those of CD. The line AB intersects the line C_tD_t at the point M. Draw a line MN through the point M and perpendicular to the plane T, to intersect CD at the point N. The line MN is the required perpendicular, and its true length is mn_r.

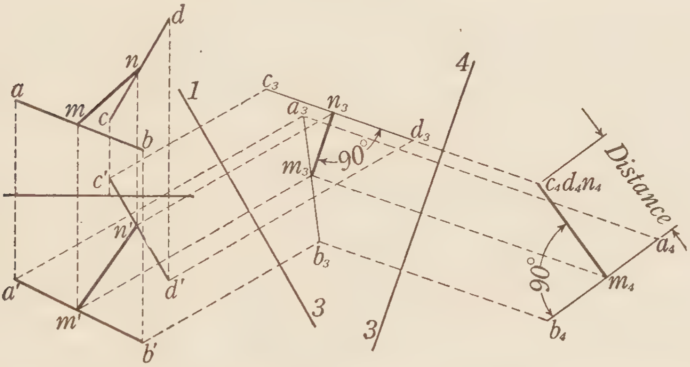


FIG. 97. Distance between two non-parallel, non-intersecting lines

Construction Two. Given the two lines AB and CD, Fig. 97. Set up the auxiliary coördinate plane 3 parallel to CD, by drawing *1–3* parallel to $c'd'$, and project the lines AB and CD upon it at a_3b_3 and c_3d_3 respectively. Set up the auxiliary plane 4 perpendicular to the line CD, by drawing *3–4* perpendicular to c_3d_3, and show the projections of the two lines upon it at a_4b_4 and c_4d_4. From n_4, which is coincident with c_4d_4, draw n_4m_4 perpendicular to a_4b_4. This is the plane-4 projection and the true length of the required perpendicular between the two given lines. The projection m_4 returns to m_3 on a_3b_3. From m_3 a perpendicular, m_3n_3, is drawn to c_3d_3, thus locating n_3. The projection m_3n_3 is perpendicular to c_3d_3 because the line CD is parallel to the plane 3, and a line perpendicular to CD shows at right angles to it in this projection. The projection m_3n_3 returns to the horizontal and vertical projections in the usual manner. The line MN is the required perpendicular.

78. Projections of solids

It has been shown in preceding articles how to represent a plane geometrical figure of any shape in any position in space, when the plane in which the figure is to be drawn is given. It is still more essential, however, that the draftsman be able to represent solid geometrical figures in whatever position may be desired in space.

Most solid objects used in engineering design are bounded either by plane geometrical surfaces of an elementary shape, such as triangles, squares, rectangles, etc., or by curved surfaces in combination with plane surfaces. The right pyramid, prism, cone, and cylinder are typical examples of the surfaces commonly used. The projections of any one of them are easily obtained when its axis is perpendicular to one of the principal coördinate planes. If the axis should be inclined to all the coördinate planes, it is necessary to apply some of the principles and methods of construction developed heretofore. A few cases are demonstrated in the following articles.

79. To construct a right pyramid with a given point as its apex and with its base, of given shape and size, in a given plane

Analysis. The axis of the pyramid will be perpendicular to the given plane, will pass through the given point (the apex), and will pierce the plane in a point which will be the center of the base of the pyramid. The specified base may be drawn about this point by the methods of Article 55. By connecting the projections of the apex to the corresponding projections of the corner points of the base and using the proper conventional lines for the invisible portions of the pyramid, the solid will be completely represented.

Construction One. *Oblique-plane method.* Given the point A as the apex of a right pyramid whose base is a regular hexagon, X inches on a side, and in the given plane S, in the first drawing of Fig. 98. Find the point P in which the line AB, drawn through the point A and perpendicular to the plane S, pierces the plane S. Revolve the plane S with the point P into the vertical plane about Ss' as an axis. The point P falls at p_r. Construct the regular hexagon $c_rd_re_rf_rg_rk_r$, X inches on a side, about p_r as a center. Counter-revolve the plane S, by either of the methods of Article 55, to its original position in space and obtain both the vertical and the horizontal projections of all the vertices of the hexagon. Join these projections in the proper order to each other and to the corresponding projections of the point A. The edges AK, KC, and KG of the pyramid are invisible in the vertical projection; the edges AD, DC, and DE are invisible in the horizontal projection; all are represented by broken lines.

Discussion. The right cone may be constructed by a method similar to that used for the pyramid, if the apex and the size and plane of the base are given. Often the center of the base of the cone, the plane and size of the base, and the length of the axis are known. In such a case the apex of the cone must be located. This may be done by drawing a line of any assumed length from the center of the base and perpendicular to the plane of the base, and finding its true length by any of the methods developed heretofore. The specified length of the axis may then be laid off on this true-length projection and the apex, thus determined, returned to the principal projections of the axis (see Article 63).

Construction Two. *Auxiliary-plane method.* Given the point A as the apex of a right pyramid whose base is a

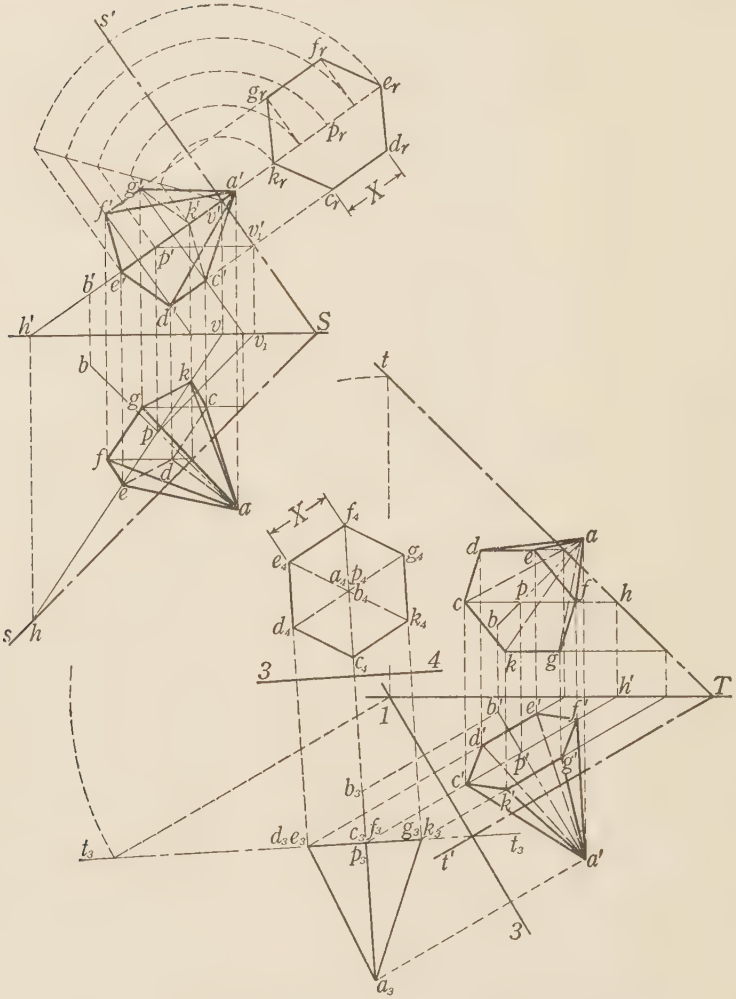


FIG. 98. Projections of a right pyramid

regular hexagon, X inches on a side, and in the plane T, in the second drawing of Fig. 98. Draw a line AB of any length from the point A and perpendicular to the plane T. This line contains the axis of the pyramid and pierces the plane T in one end of the axis. Set up the auxiliary coördinate plane 3 parallel to the line AB and hence perpendicular to the plane T, by drawing *0–3* parallel to ab. The line AB projects upon the plane 3 as a_3b_3. The trace of the plane T on plane 3 is t_3t_3, which, being the edgewise view of the plane, contains the plane-3 projection of the base of the pyramid. The projection a_3b_3 intersects t_3t_3 at p_3, the plane-3 projection of the center of the base of the pyramid. Set up plane 4 perpendicular to the axis AP by drawing *3–4* perpendicular to a_3b_3. The line APB projects upon plane 4 at $a_4p_4b_4$. The base of the pyramid projects as a regular hexagon in true size on this plane and may be drawn about p_4 as a center in the usual way, at $c_4d_4e_4f_4g_4k_4$. Return this hexagonal base to its plane-3 projection as the straight line e_3k_3 in the trace t_3t_3, and join the projection of each point thus obtained to a_3. The H and V projections of the pyramid are obtained from the plane-3 projection in the usual way and the invisible lines indicated as in Construction One. If the views on planes 3 and 4 are for construction purposes only, the invisible lines in these views need not be determined.

Discussion. Auxiliary coördinate planes versus oblique planes in projecting solids and surfaces. In representing solids and surfaces, such as prisms, cylinders, and the like, where two bases have to be shown, the auxiliary-coördinate-plane method offers many advantages over the oblique-plane method. Views on planes 3 and 4 may be made the simplest views of the object that it is possible to obtain and are, therefore, related to each other exactly as the horizontal and vertical views would be were the object placed with its axis perpendicular to one of the principal coördinate planes. Ground lines in auxiliary views are more or less arbitrary guides in construction and may be treated as mere reference lines, or omitted altogether, since the relations between the various points and lines on a solid object can be maintained without them. They cannot be omitted when the planes are represented by conventional traces. In developing surfaces (see Article 107) the auxiliary views have marked advantages over the regular views when the objects involved are inclined to the coördinate planes.

80. To construct a right prism with bases of given shape and size and with a given line of specified length as an axis

Analysis. If an auxiliary plane 3 is set up parallel to the axis of the prism, that is, perpendicular to the planes of its bases, the prism will project upon this plane as a rectangle in outline equal in length to the specified length of the axis. If plane 4 is set up perpendicular to the axis, the prism will project upon it as a polygon of the true size and shape of the base. From these auxiliary projections both the horizontal and the vertical projections may be found.

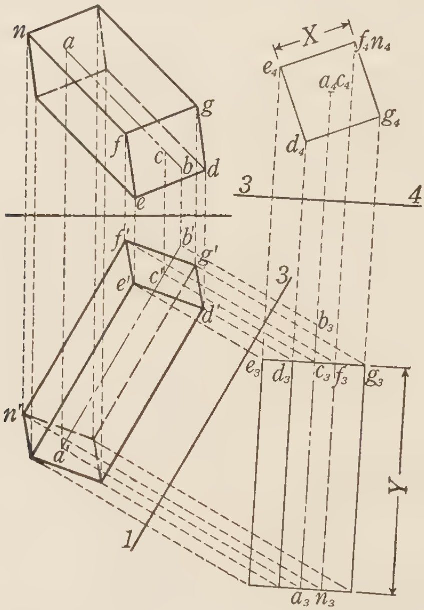


FIG. 99. Projections of a right prism

Construction. Given the point A as the center of one square base of a right prism whose axis, Y inches in length, lies in AB and whose base is X inches on a side, Fig. 99. Set up an auxiliary plane 3 parallel to AB by drawing the ground line *1–3* parallel to $a'b'$, and project AB upon this plane at a_3b_3. Both bases of the prism show upon this plane as straight lines perpendicular to a_3b_3, since the planes of the bases are perpendicular to AB and therefore show edgewise on plane 3. Lay off the specified length of the axis from

a_3, on a_3b_3, to obtain c_3, the center point of the plane-3 projection of the second base. Set up plane 4 perpendicular to the line AB by drawing the ground line *3–4* perpendicular to a_3b_3 and project the axis AC upon it at a_4c_4. The bases and all the sides of the prism project upon plane 4 as the square $d_4e_4f_4g_4$, X inches on a side, with a_4c_4 as the center. Through c_3 draw a line perpendicular to a_3b_3 and return $d_4e_4f_4g_4$ to this line as $d_3e_3f_3g_3$. Through a_3 draw a line perpendicular to a_3b_3 and project the second base of the prism upon it by means of the same projecting lines used in obtaining $d_3e_3f_3g_3$ from the plane-4 projection. Connect the plane-3 projections of the corresponding points of the two bases to each other. Construct the horizontal and vertical projections of the prism in the usual manner and determine the invisible lines in each of these two projections by considering the bases closed and all bounding surfaces opaque.

Discussion. The practical or direct method employed in drawing the square prism is applicable to a cylinder, cone, pyramid, or any other object.

Nothing has been said heretofore about the direction of the lines forming the bases of geometrical solids in their respective planes. Some practical consideration of design always controls this feature of the drawing. Usually an edge line or diagonal of the base will be specified as being parallel to one of the coördinate planes, in which case a line should be drawn in H and V projections to satisfy the specification; then its projections on planes 3 and 4, or its revolved position if the oblique-plane method is used, may be employed to direct the drawing of the base in question.

In practical work there will be no traces of the plane of the base to direct the drawing of the projections of the axis. The direction of this essential line must be determined from lines already located on the drawing, or it must be expressed in terms of the angles the axis makes with the coördinate planes (see Article 111), or in some other way made definite.

CHAPTER V

SURFACES: THEIR GENERATION, REPRESENTATION, INTERSECTIONS, DEVELOPMENTS, AND USES

81. Lines

Before introducing the analytical study of surfaces and their uses in geometrical constructions and in engineering drawing, it will be well to repeat in part, and then to extend, the conceptions already formed concerning the properties of the line.

A line is the geometrical magnitude formed or generated by a point moving through space. If the point moves without change of direction, the line generated is a straight line. If the point moves so that its direction changes continuously, the line generated is a curved line. It is customary to define the nature of the motion of the generating point with respect to a set of coördinate planes, or, in other words, the character and position of the line generated, by the use of algebraic equations, such, for example, as $x^2 + y^2 = b^2$ or $xy = c$. By assigning numerical values to one variable in the equation and assuming some value for the constant, corresponding values of the second variable may be obtained and the line plotted on coördinate paper.

Many of the commonly used curved lines are given specific names by which they are more generally recognized than by the formulas from which they are plotted. Some of these are the circle, the ellipse, the spiral, and the helix.

All the points of a straight line are contained in one plane. If all the points of a curved line are contained in one plane, the curve is known as a plane curve or a two-dimensional curve. All other curves are called space curves or three-dimensional curves. Plane curves are called curves of single curvature, while space curves are known as curves of double curvature.

Let it be assumed that the point A_0 in the first drawing of Fig. 100 moves to the position A_1, then to A_2, and so on through successive positions to A_n, all the points being in the plane of the paper, and let it be assumed, further, that the distances between A_0 and A_1, A_1 and A_2, etc., are less than any assigned value; that is, if the distances are made smaller, each point will coincide with its neighbor; then the points are said to be consecutive and the infinitesimally small segments A_0A_1, A_1A_2, A_2A_3, etc., of the line, numbered *1*, *2*, *3*, etc. in the figure, are called elements of the line.

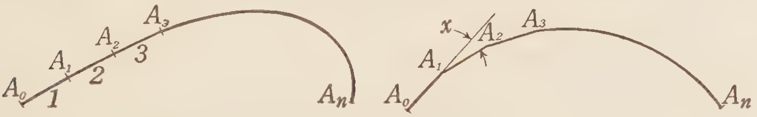


FIG. 100. Consecutive points; elements of a line

If, in the second drawing of Fig. 100, the direction of element *2* of the plane curve $A_0A_1A_2 \cdots A_n$ differs from that of element *1* by a small angle α, and the same change in direction takes place for each succeeding element, the curve is a circle. All other plane curves have varying rates of curvature as the generating point marks out their shapes in its movement.

If a point moves in such a manner that no three consecutive elements are contained in the same plane, the curve generated is one of double curvature. The helix is a good example of this class of curves.

82. Tangents and normals to curves

A line containing only one of the elements of a curve is called a tangent to the curve. This means, in the case of a plane curve, that the tangent must lie wholly in the plane of the curve. Curves, like straight lines and other magnitudes, are represented in descriptive geometry by their projections. A straight line tangent to a curve in space will have its projections tangent to or coincident with the corresponding projections of the curve, and, conversely, if the corresponding projections of a straight line and a curve, or of two curves, are tangent, the lines are tangent. A line perpendicular to a tangent to a plane curve at the point of tangency (the point of coincidence of two approaching consecutive points) and lying in the plane of the curve, is called a normal to the curve.

83. Generation of surfaces

A definable surface is a geometrical magnitude that is generated by a straight or curved line moving in space according to some law. Since there are many kinds of curved lines and since the motion of any one of them may be in the form of a translation or a revolution, it follows that there are many kinds of surfaces. Some of the more familiar are the plane, the cone, the cylinder, the sphere, and the torus.

The line used to generate any surface is called the generatrix. Its motion may be controlled by one, two, or more fixed lines, called linear directors, about which it may revolve, along which it may slide, or by which it may in other ways be regulated in its movement. For example, the plane is generated by a line sliding along two parallel or intersecting straight lines; a sphere is generated by a semicircle revolving about its diameter; a helical convolute is generated by a straight line moving tangent to a helix. The generatrix may also be controlled in its movement by a plane to which it remains parallel or inclined at a constant angle, or by a plane and one or two straight or curved lines in combination. A plane used in this manner is called a plane director.

Let it be assumed in the first drawing of Fig. 101 that the line A_0B_0 has moved to the successive positions A_1B_1, A_2B_2, etc., to generate a portion of a cylindrical surface, and let it be assumed, further, that the distance between any two successive positions of the line is less than any assignable value;

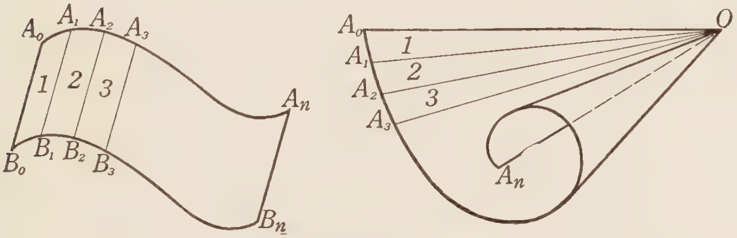


FIG. 101. Consecutive lines and elements of surface

then A_0B_0, A_1B_1, A_2B_2, $\cdots$, A_nB_n are known as consecutive elements, and each small area contained between any two consecutive elements, such as $A_0B_0B_1A_1$, for instance, is called an element of surface. In the case of the conical surface in the second drawing of Fig. 101, the triangular-shaped area OA_0A_1 is an element of surface. Fig. 102 shows several elements of surface of the sphere and the torus.

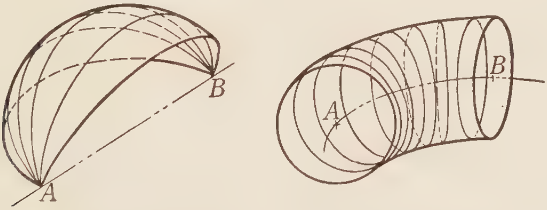


FIG. 102. Elements of a sphere and torus

In those cases in which a straight line is used as the generatrix, the line and hence the surface are considered infinite in extent for all theoretical discussions, but in practical applications it is necessary to restrict the surface to definite limitations. This introduces the concept of bases and sections, particularly in connection with the so-called closed surfaces. Although all closed geometrical figures are treated as surfaces in this chapter, any one of them may be considered a solid and the analyses or discussions applied to it without material alteration.

84. Classification of surfaces

Surfaces are classified according to the kinds of lines used as generatrices. If the generatrix is a straight line, any surface that can be formed by its movement is called a ruled surface. A straight line lying wholly in the generated surface can be drawn through any chosen point on such a surface. If the generatrix is a curved line and if the surface formed by its movement cannot be generated by a straight-line generatrix — in other words, if the surface is curved in every direction — it is called a double-curved surface. Straight lines may not be drawn on such a surface. These two major divisions, into one or another of which all surfaces are classified, may be further subdivided and named.

There are three kinds of ruled surfaces, namely, planes, single-curved surfaces, and warped surfaces. Single-curved surfaces are further classified as cones, cylinders, and convolutes. Warped surfaces can be subdivided into several general types such as conoids, hyperbolic paraboloids, cylindroids, helicoids, and hyperboloids of revolution (see illustrations in Fig. 110). Many other ruled surfaces can be generated, but few type names have been evolved.

Double-curved surfaces of any useful character fall mostly into one class called surfaces of revolution. The sphere, the prolate and oblate ellipsoids of revolution (spheroids), and the torus are well-known examples in this class. The serpentine is illustrative of the class of double-curved surfaces not generated by the revolution method.

Surfaces may also be classified on the basis of the character of the movement of the generatrix. If the motion of the generatrix is one of sliding or, more correctly, translation, the surfaces generated are called surfaces of translation. If the motion is that of revolution, implying, of course, that there must be a straight-line axis, the surfaces generated are called surfaces of revolution. Some surfaces may be generated by either the method of translation or that of revolution. The circular cylinder is a good example. It should be noted here, also, that some surfaces may be generated by using either of two, and sometimes more, generatrices. The example of the cylinder just cited is a case in point. Because of these facts and others not mentioned, the divisions in the classification of surfaces given in this paragraph must not be considered as mutually exclusive.

A third classification of surfaces of a useful kind is that based on the characteristic of developability, that is, whether they can be rolled out flat without distortion of any of their elements. Single-curved surfaces are the only theoretically developable surfaces.

The above classifications of surfaces are summarized in the following tabular form:

TABLE I. CLASSIFICATION OF SURFACES

I. Ruled
    A. Plane
    B. Single-curved (developable)
        1. Cone
        2. Cylinder
        3. Convolute
    C. Warped
        1. Conoid — right and oblique
        2. Cylindroid
        3. Helicoid — right and oblique
        4. Hyperbolic paraboloid
        5. Cow's horn
        6. Hyperboloid of revolution of one sheet (revolved about conjugate axis)

- *D*. Surfaces of revolution
    1. Circular cone
    2. Circular cylinder
    3. Hyperboloid of revolution of one sheet (revolved about conjugate axis)

II. Double-curved

- *A*. Surfaces of revolution
    1. Sphere
    2. Spheroid — oblate or prolate ellipsoid of revolution
    3. Ovoid
    4. Torus
    5. Paraboloid
        - *a*. Revolved about directrix
        - *b*. Revolved about axis
    6. Hyperboloid of revolution of two sheets (revolved about transverse axis)
    7. General (any plane-curve generatrix)
- *B*. General
    1. Ellipsoid
    2. Serpentine

85. Representation of surfaces

The projections of the contour or bounding outlines of a surface, as they would be seen from the infinite point of sight required for each orthographic view, are usually sufficient to represent the surface on a coördinate plane. These contour, or extreme, outlines will generally be different bounding lines of the surface for each view. In the case of plane figures, they will be the same lines for all views. Sometimes additional primal lines of a surface, or a cross section through it, must be projected to aid in completely representing it. Generally one or more significant points on the surface or on the axis are lettered, and the surface is then referred to by means of these letters by such expressions as "the cone A," "the cylinder whose axis is AB," "the sphere whose center is C," and so on. In the case of irregular or unnamed surfaces, more letters must be used to prevent ambiguity in differentiating one part of the surface from another.

86. Single-curved surfaces

A single-curved surface is generated by a straight line moving so that it always touches a given curved-line director and either passes through a fixed point (cone), remains parallel to a fixed straight line (cylinder), or moves tangent to some space curve (convolute).

No other single surface, except the plane, is so widely known and used as either the cone or the cylinder. Both are developable ruled surfaces. Each may occur in the form of an unclosed scroll, as in Figs. 100 and 101, often termed cylindrical and conical surfaces, or they may, and usually do, appear in the form of closed surfaces like those shown in Figs. 103 to 108, inclusive. In using the terms "cone" and "cylinder," it is customary to include in the reference only the closed forms, as is done hereafter in this text. No difficulty is experienced in applying the discussions to the unclosed forms of the two surfaces. The properties of the cone are usually developed before those of the cylinder in treatises on descriptive geometry, because of the fact that the cylinder is the limiting condition of the cone as its apex is moved to infinity.

Convolutes are developable surfaces. No three consecutive positions of their straight-line generatrices lie in a plane or intersect in a point. The helical convolute is the most common of this class of surfaces (see Fig. 109).

Each of the single-curved surfaces mentioned above is discussed in greater detail in the following articles.

87. Cones

The cone is a ruled surface generated by a straight line moving so that it constantly touches a given curve and passes through a fixed point. The fixed point is called the apex of the cone; the guiding curve is called the director; and the moving straight line, the generatrix. The various successive positions of the generatrix are known as the elements of the cone. Since the moving line which generates the cone is infinite in length, the surface itself is infinite in extent in both directions from the apex. The two connected surfaces are called the *nappes* of the cone. Usually one nappe only is considered in any discussion.

Bases of cones. It is evident that only a portion of any generated cone may be used in practical work. This portion, however, is still designated the "cone," and its properties remain identical with those of the infinite cone, except that its elements all terminate in a finite curve. This curve is spoken of as the base of the cone; and if a plane curve, the plane in which it lies is called the plane of the base. The portion of the cone between two plane-curve bases is termed a *frustum,* even when the planes of the two bases are not parallel. Any other curves which may be cut from a cone by planes passed at will are called plane sections. If the cutting planes are parallel, the sections are similar.

Right and oblique cones. Considering any given cone: If a plane can be passed in such a way that it will intersect the cone in a circle, and if the apex of the cone is a point in the line drawn through the center of this circle and perpendicular to its plane, then the cone is a circular cone. If the elements are all made to terminate in the circle, the cone is a right circular cone. If the elements do not terminate in a circle, the cone is an oblique circular cone. If any given cone is not a circular cone and a plane can be passed through it in such a way that the intersection will be an ellipse, and if the apex is a point in the line drawn through the center of this ellipse and perpendicular to its plane, then the cone is an elliptical cone. If the elements are all made to terminate in the ellipse, the cone is a right elliptical cone. If the elements do not terminate in an ellipse, the cone is an oblique elliptical cone. Fig. 103 shows the H and V projections of a right circular and a right elliptical cone after they have been placed in the most favorable position, with respect to the coördinate planes, for showing their true shape and size. The apex of a cone is usually lettered and the cone itself is regularly referred to by means of this letter. Other right cones may be obtained by choosing differently shaped plane curves as bases. These plane curves must, however, have at least two lines of symmetry. The apexes of such cones must lie in perpendiculars to the planes of the base curves at the points of intersection of the two lines of symmetry.

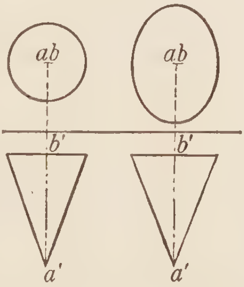


FIG. 103. Right cones

Axes of cones. The line drawn through the apex and the center of the base of each of the right cones of Fig. 103 is called the axis of the cone. Any given cone which can be reduced to a right cone by means of a new base plane has an axis. Other cones do not have axes. Any cone from which a circle or an ellipse can be cut by a plane has an axis, because it can be reduced to a right cone. The axis is used in specifying the obliquity of the plane of the base of the cone and in giving the general relation of the cone to the coördinate planes. Except in circular cones, the axis does not make equal angles with the elements. If the base of a cone is not in a plane perpendicular to its axis, the cone is said to be an oblique cone. Thus we may have such expressions as "an oblique circular cone with an elliptical base" or "an oblique elliptical cone with a circular base." The mere changing of the inclination of the base plane to the axis, or the axis of a cone with respect to the coördinate planes, does not, of course, change the character or properties of the cone. In case the axis is not shown in a given instance, it may prove a tedious process to locate it accurately.

Fig. 104 shows the two right cones of Fig. 103 with their axes inclined to the coördinate planes. Since the axis of each cone and the shape and the size of the base of each were given, and since it was known that the cones were right cones, the projections of the base of each were obtained by means of the oblique planes S and T, as demonstrated in Article 77. The projections of the axes of the two cones pass through the centers of the corresponding projections of the bases. This would not have occurred were the cones not right cones, as will be explained in the next paragraph.

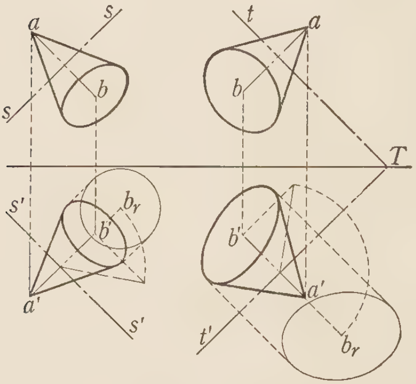


FIG. 104. Right cones, axes oblique

Lines of centers. The cone of Fig. 105 is an oblique elliptical cone with a circular base. In this case the line AC, joining the apex to the center of the base, is not the axis of the cone. It does pass through the centers of all the circles that are cut from the cone by planes passed parallel to the plane of the base, such, for example, as plane R. The line AC, therefore, is called a line of centers. If a plane T, not parallel to plane R, be found which is perpendicular to the principal

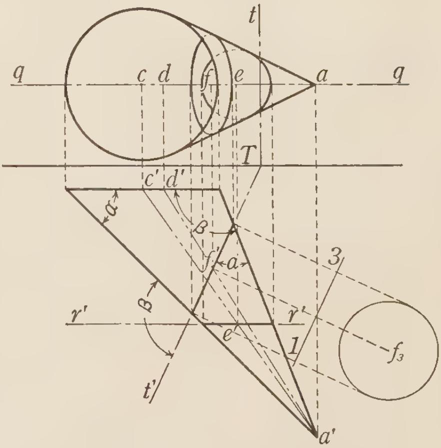


FIG. 105. Axis of oblique elliptical cone

meridian plane, Q, and that makes the same angles, α and β, with the elements cut from the cone by the plane Q that plane R makes with these elements, then the section cut from the cone by plane T will be a circle whose center is F. The line AD, drawn through the point F, will be the line of centers of all the circles cut from the cone by planes passed parallel to the plane T. The axis of the cone is a line, AK, not shown on the drawing, that bisects the angle CAD. A plane perpendicular to AK cuts a right section from the cone.

Assuming elements on cones. It is essential that the apex, a projected base, and the extreme elements of a cone be shown in order to represent the surface. If these three things have been drawn in two or more projections, it is possible to determine an element of the cone by selecting a point on the base curve and joining it to the apex by a straight line. Fig. 106 shows an elliptical cone on which the element AM has been chosen in accordance with this principle.

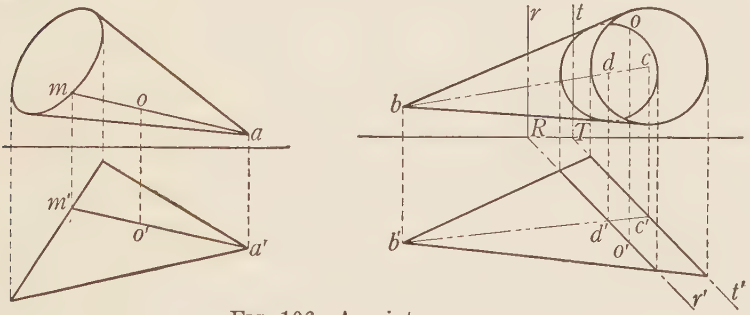
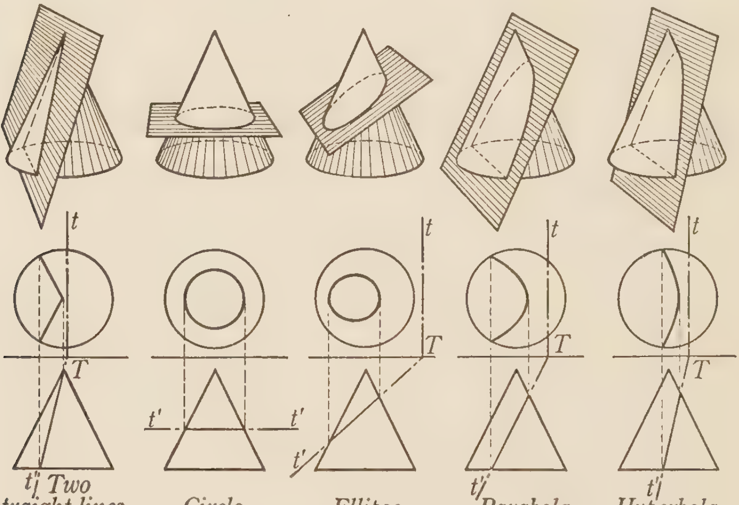


FIG. 106. A point on a cone

A light line is often used to distinguish the projections of elements, chosen for purposes of construction, from the contour elements originally projected. In some cases it is desirable to use a dash line for those elements which are invisible in any view. In determining whether an element is invisible or not, it is the practice to assume that the surface is opaque and that the base is open, or, in other words, that there is no covering surface within the base curve.

Assuming points on cones. Since an element of a cone is a straight line, a point may be selected on the cone by taking it on some element of the cone. The point O of the first drawing of Fig. 106, being on the element AM, is on the cone.

A second method of selecting a point on a cone may be used to advantage in the case of a cone so related to the



Two straight lines — Circle — Ellipse — Parabola — Hyperbola

FIG. 107. Conic sections

coördinate planes that one projection of its base is a circle and the other a straight line. The plane of the base of such a cone is a projecting plane (see plane T in the second drawing of Fig. 106). A plane R, passed parallel to plane T, cuts a section from the cone similar to the base. In the figure this section, which is an ellipse, projects as a circle on the horizontal plane with its center at *d*. Any chosen point on this curve, such as O, is a point on the cone.

Bases of cones in the coördinate planes. Once the elements of a cone have been determined, they may be extended until they pierce any one of the coördinate planes. A smooth curve joining the H or V piercing points of a considerable number of well-chosen elements will constitute a new base in the H or V plane which is very useful in solving problems dealing with the cone. One projection of such a base is always in the horizontal ground line.

Conic sections. Planes may be passed through a circular cone in such ways as to cut from it four sections called *conic sections.* These sections are the circle, ellipse, parabola, and hyperbola. The position of the cutting plane for each section is shown in Fig. 107. Planes may also cut the cone in two straight lines, in one straight line, or in a point. The plane which contains only one element of the cone is tangent to the cone. The plane which cuts a parabola from the cone must be parallel to some tangent plane.

88. Cylinders

The cylinder is a single-curved surface that is generated by a straight line so moving that it constantly touches a given curve and remains parallel to a given straight-line director. The given curve is called the director and the moving line the generatrix. The cylinder is the special case of a cone with its apex at infinity and, like the cone, may be either a closed or an open surface. The terms "element," "axis," "base," and "section" have the same significance as when used in connection with the cone.

A right section of a cylinder is obtained by passing a plane perpendicular to its elements. If this section is a circle, the cylinder is a circular cylinder; if the right section is an ellipse, the cylinder is an elliptical cylinder. If the elements terminate in two right sections, the cylinder is called a right cylinder; if they terminate in two sections that are not right sections, the term "oblique" is used in describing the cylinder. A cylinder may be a right cylinder with respect to one base and oblique with respect to the other. Since the generatrix is a line of infinite length, the cylinder is considered infinite in extent for theoretical purposes, but for practical uses it is limited by two bases that are a finite distance apart.

A cylinder is represented by the projections of at least one base and the projections of the contour or extreme elements in each view, as shown in Fig. 108. An axis, or line of centers, may be shown, when possible, to assist in referring to the cylinder or to aid in constructions.

An element of the cylinder may be assumed in the same manner as on the cone. Any point on the base section may be chosen and its projections located. The element will pass through this point and be parallel to the contour elements, thus fixing the projections of the element in both direction and location (see AN, Fig. 108). Any point O on the element AN is on the cylinder.

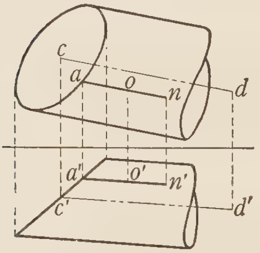


FIG. 108. A point on a cylinder

It is often desirable to extend a given cylinder until it intersects one of the coördinate planes. This may be done by locating a considerable number of elements around the cylinder and producing them one after another to pierce the desired coördinate plane. A smooth curve joining these piercing points forms a new base section which is very useful in solving problems involving the cylinder. One projection of such a base in the H or V plane is always in the horizontal ground line.

89. Convolutes

Fig. 109 shows a helical convolute of one complete turn. Each element is tangent to the helical director and intersects its two adjacent elements in separate points. Only two elements can be in the same plane. The surface is developable. It was formerly used in making screw conveyors and in one system of skew bevel gears known as "Olivier spiraloids." The helical convolute is often called the developable helicoid. It should not be confused with the warped surfaces known as helicoids, which are generated in an entirely different manner.*

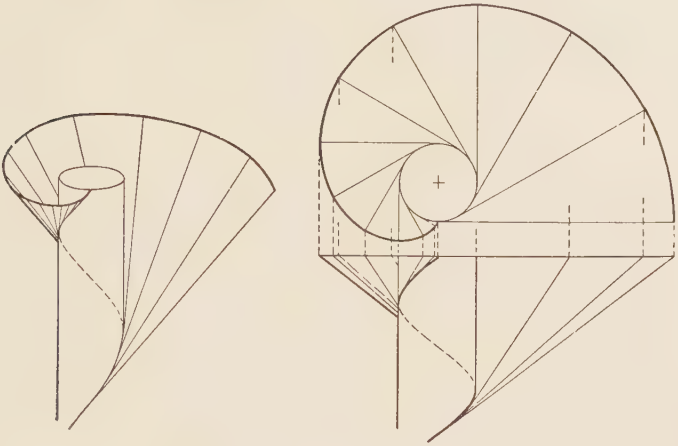


FIG. 109. Helical convolute

90. Warped surfaces

A warped surface is generated by moving a straight line so that no two of its consecutive positions are in the same plane. Two straight or curved lines along which the generatrix may slide, and a plane to which the generatrix remains parallel, constitute one set of a number of combinations of linear, plane, or curved-surface directors which may be specified in fixing the motion of the generatrix. Since the generatrix is a straight line, the surface is a ruled surface. The surface is not developable, because no two adjacent elements intersect. In representing a warped surface, it is necessary to project the linear directors and enough elements of the surface to give an adequate conception of its shape. Fig. 110 shows five typical warped surfaces. They are characterized briefly in the following paragraphs.

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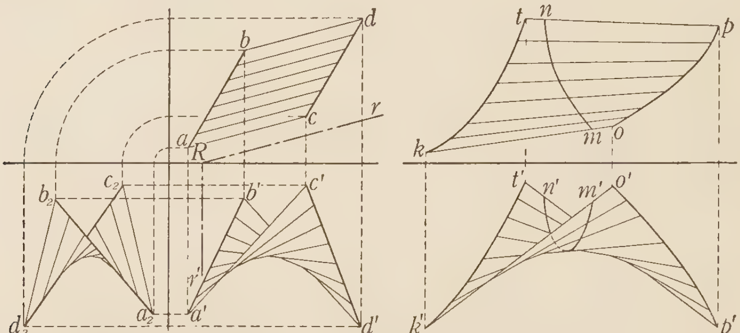
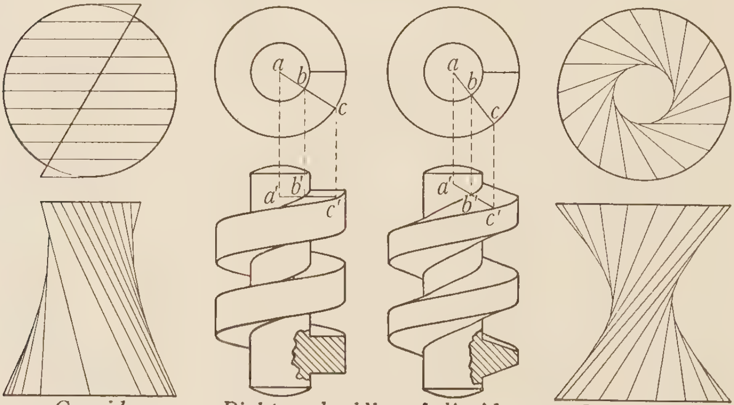
* For a further discussion of the convolute and the method of developing the surface, see Church and Bartlett's "Elements of Descriptive Geometry" (American Book Company).

The first surface in Fig. 110 is typical of a group of warped surfaces called hyperbolic paraboloids. These surfaces are formed by moving a straight-line generatrix along two non-intersecting straight-line directors and parallel to a plane director, in this case the H projecting plane R, or along three non-intersecting straight lines. Any plane section other than those through a linear director or a gen-



Hyperbolic paraboloid

Cylindroid



Conoid

Right and oblique helicoid

Hyperboloid

FIG. 110. Warped surfaces

eratrix are hyperbolas or parabolas; hence the name of the surface. This surface is of interest principally from the mathematical point of view.

The second surface of Fig. 110 is an example of a class of warped surfaces called cylindroids. These surfaces are formed by moving a straight-line generatrix along two curves and parallel to a plane or along three curves. The cylindroids, like the hyperbolic paraboloids, find little application in practical construction work.

The warped surface shown in the third drawing of Fig. 110 is typical of a group called conoids. They are formed by moving a straight-line generatrix along one straight-line and one curved-line director and parallel to a plane director, in this case, the vertical plane. They are used in building construction as transition sections from a curved surface, such as the roof of a vault, to a rectangular opening in a plane wall.

The fourth drawing of Fig. 110 represents the warped surfaces called helicoids. They are generated by a straight line moving so as to touch a helix constantly and intersect the axis of the helix at a constant angle. This surface is widely used. When the generatrix is perpendicular to the axis of the helix, the right helicoid is generated. This is the surface of the well-known square thread. When the generatrix is inclined to the axis of the helix, the oblique helicoid is formed. This is the surface of the theoretical V screw-thread. This surface also finds application in the design of circular staircases and chutes.

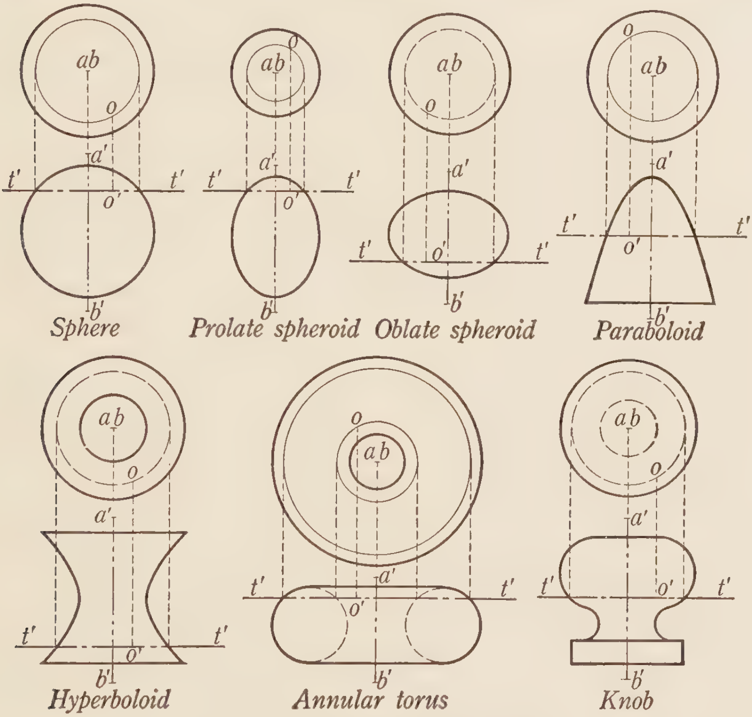
The last drawing of Fig. 110 is the only warped surface that is also a surface of revolution. It is known as a hyperboloid of revolution and is generated by revolving a straight line about an axis not in the same plane. It can also be generated by revolving a hyperbola about its conjugate axis. It is represented by the projections of two circular sections cut from the surface by planes perpendicular to the axis of revolution, the projections of the smallest circular section, called the gorge circle, the projection of meridian hyperbolas on the coördinate plane to which they are parallel, and the projections of as many straight-line elements as are necessary to satisfactorily delineate the surface. The surface finds its greatest application in the making of skew bevel gears.

91. Surfaces of revolution

Surfaces of revolution are generated by revolving a straight or curved line about a rectilinear axis. They may, therefore, be either ruled surfaces or surfaces of double curvature. The moving line is called the generatrix, and the line about which it revolves is called the axis of revolution. If the generatrix is a straight line parallel to the axis of revolution, the surface generated is a circular cylinder. If the generatrix is a straight line intersecting the axis of revolution, the surface generated is a circular cone. If the generatrix is a straight line not in the plane of the axis of revolution, a hyperboloid of revolution is generated. The two first-named surfaces are single-curved ruled surfaces. They have been discussed in preceding articles. The hyperboloid of revolution is the only warped surface of revolution.

If the generatrix is a curve, such surfaces as the sphere, torus, and the ellipsoids, paraboloids, and hyperboloids of revolution are obtained. If a circle is revolved about its diameter as an axis, the surface generated is a sphere. If an ellipse is revolved about its major axis, the surface generated is a prolate ellipsoid of revolution (prolate spheroid); if an ellipse is revolved about its minor axis, the surface generated is an oblate ellipsoid of revolution (oblate spheroid). If a circle, an ellipse, or any other smooth closed curve is revolved about an axis outside the curve, the surfaces generated are tori or rings. If the circle is used as the generating curve, the annular torus is formed. Other surfaces of revolution may be generated by using differently shaped curves as generatrices, but few type names have been evolved to describe them. Several surfaces of revolution are shown in Fig. 111.

If a plane is passed perpendicular to the axis of any surface of revolution at any point, it will cut a circle from the


Sphere Prolate spheroid Oblate spheroid Paraboloid

Hyperboloid Annular torus Knob

FIG. 111. Surfaces of revolution

surface, called a right section. Planes passed through the axis of revolution cut curves from the surface which are alike and identical in shape with the generatrix. They are called meridian sections or elements.

Since a surface of revolution is usually drawn with its axis perpendicular to a coördinate plane, it may be completely represented by the projection of one meridian section on any coördinate plane to which such a section may be taken parallel and by the projections, on the plane to which the axis is perpendicular, of the largest and smallest right-section circles, and by the projections of such other right sections at points of change in curvature in the generatrix as may be necessary.

A point may be assumed most readily on a surface of revolution by first obtaining the projections of a circle that is cut from the surface by a right-section plane and then selecting a point on this circle. The point O on each of the surfaces of revolution of Fig. 111 has been obtained by passing the plane T perpendicular to the axis of revolution, AB, of each surface, projecting the circle cut from each surface upon the two principal coördinate planes, and then selecting the projections of the point O on the corresponding projections of the circle.

It should be noted that the plane passed perpendicular to the axis of revolution in each case of Fig. 111 is a projecting plane. This makes it possible to draw one of the tangents at the point O on each surface without complicated constructions by putting it in one of these section planes and tangent to the circle cut from the surface.

Surfaces of revolution, like all other surfaces, are considered opaque; hence a line on the surface which is hidden from the observer in any view by any portion of the surface must be shown in the usual broken-line convention. However, when section planes are passed through a surface simply for construction purposes, the curves cut from the surface may be shown as light full lines.

92. Intersection of surfaces

The *intersection* of two surfaces is defined as the line containing all points common to the surfaces. It is the line that joins together the points in which the elements or lines of the one surface pierce the other surface. The intersection may be a straight line, a plane curve, or a space curve. There may be two distinct lines of intersection between two surfaces, as would be the case if a small cylinder were made to penetrate both sides of a larger cylinder with the center lines of the cylinders intersecting each other. Sometimes several distinct surfaces are dealt with in the same problem, as would be the case in finding the intersection of two prisms, each having three or more plane faces, or in the case of the intersection of a prism and a cylinder or cone where the faces of the prism intersect the second surface in several distinct but connected lines.

The problem of finding the line of intersection of any two surfaces resolves itself into a succession of similar steps, in which are found the points where a number of carefully determined elements or plane curves of the one given surface pierce the second surface, these points then being joined in the proper order by a straight line or a smooth curve. The approach to the general application of the theory to this problem is best made by means of the simple case of two intersecting plane surfaces.

93. To find the line of intersection of two limited plane surfaces

Analysis. Let it be required to find the line of intersection of the triangle EFG and the parallelogram ABCD, Fig. 112.

The line of intersection is the straight line joining the points in which the edge lines of either surface pierce the other. An inspection of the figure indicates that the required line of intersection will be limited in its extent by the sides of the triangle rather than by the sides of the parallelogram; hence the points in which the lines FG and GE pierce the plane ABCD should be found.

Construction One. It was shown in Construction Two of Article 52 how the point in which a line pierces a plane may be found by a short-cut method when the plane is limited in extent by definite bounding lines, as in the case of the triangle and parallelogram of Fig. 112, and when the projections of these lines are used instead of traces to represent the plane. This method of finding the point in which a line pierces a plane is the one generally used in engineering drawing.

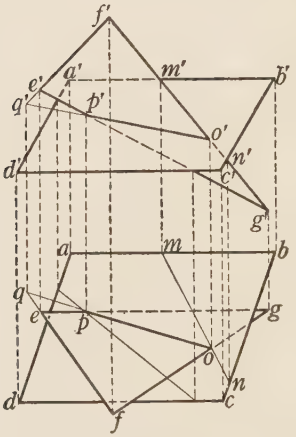


Fig. 112. Intersection of plane surfaces

Assume that a V projecting plane has been passed through the line FG, although not shown in the drawing. Find the line MN which this plane cuts from the parallelogram. The point in which FG pierces the parallelogram must lie in this line. The line FG intersects the line MN at the point O, the required piercing point of FG in ABCD. Similarly, the line GE pierces the parallelogram at the point P. The line OP is the required line of intersection. The line FE pierces the plane of the parallelogram at the point Q, which, it should be noted, lies outside the limits of both the triangle and the parallelogram, but on the line OP extended.

Discussion. Surfaces such as those of Fig. 112 are assumed to be opaque; therefore, when they intersect or are placed so that one conceals a part of the other from the observer, it is necessary to indicate the invisible portions in each view by the standard broken-line convention. This has been done in both views of Fig. 112. It should be observed that the line OP is a dividing line between visible and invisible portions of the triangle, and that no other means of separating these portions is available than that of obtaining the line of intersection of the two surfaces.

Construction Two. The principles applied in Fig. 112 to the problem of finding the line of intersection of two plane surfaces have been applied in Fig. 113 to the case of a plane triangular surface intersecting two other plane surfaces, which in turn intersect to form a V-shaped "trough." An inspection of the surfaces suggests that there are two lines of intersection that must be found. These two lines must intersect at a common point, which is the point in which the line EK pierces the triangular surface. This point is found, as in the preceding construction, by passing an H projecting plane, not shown in the figure, through the line EK, thus cutting the line MN from the triangle. The point O, common to the lines MN and EK, is the required piercing point. Similarly, the edge line FG is found to pierce the triangular surface at the point P. The line DL pierces the plane of the triangle at the point Q, which lies outside the limits of the triangle. The line joining the points O and P is one of the required lines of intersection, while the line joining the points O and Q, and terminated at the point R, to keep it within the limits of the triangle, is the second of the required lines of intersection. The invisible portions of the surfaces are shown in the usual dash-line convention.

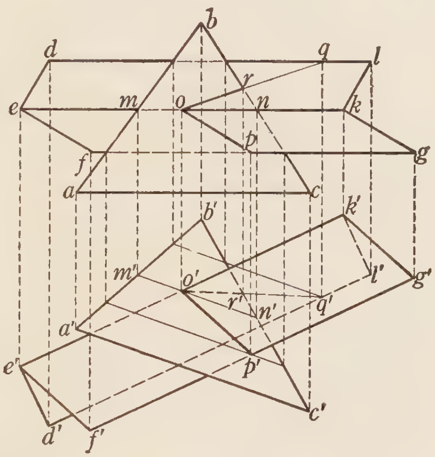


FIG. 113. Intersection of plane surfaces

Discussion. If it could have been foreseen that the line DL would pierce the plane of the triangle outside the boundaries of the triangle, the point R, in which the line BC pierces the face DLKE, would have been found directly.

94. To find the lines of intersection of prisms and pyramids

In Fig. 114 the piercing-point method of finding the line of intersection of two surfaces has been applied to the case of two intersecting prisms, and in Fig. 115 to the case of a prism intersecting a triangular pyramid. In both instances the given rectangular prism is perpendicular to the horizontal plane, thus making it possible to pass H projecting planes through each of its faces and to determine some of the points on the lines of intersection directly. Such points are M, N, O, and P in Fig. 114. These are the points in which the edge lines of the triangular prism pierce the faces of the square prism and are obtained by the short-cut method of finding the point in which a line pierces a projecting plane (see Construction Two, Article 52). The points G, J, K, and L on the required lines of intersection are obtained by means of the H projecting plane through the face EFFE of the square prism. This plane intersects the triangular prism in the triangle RNS, which contains the points G and

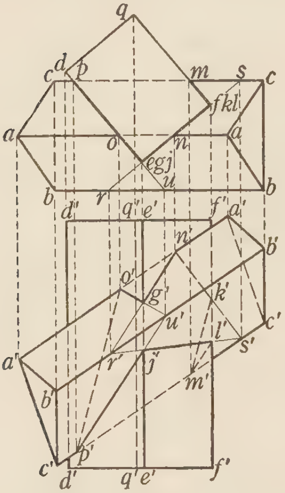


FIG. 114. Intersection of plane surfaces

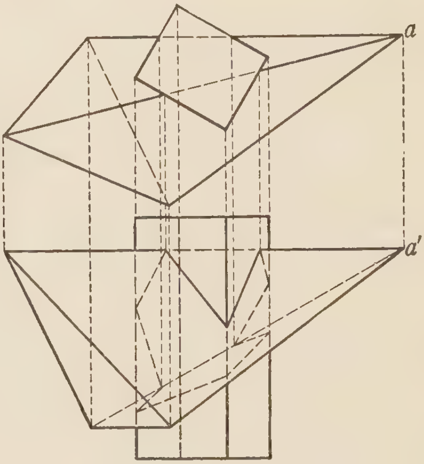


FIG. 115. Intersection of plane surfaces

J, in which EE pierces two faces of the triangular prism, and the points K and L, in which FF pierces two faces of the same prism. When all these piercing points, M, N, O, P, G, J, K, and L, are joined in the proper order, as shown in the construction, they determine the continuous but irregular line of intersection of the two prisms.

The lines of intersection of the various faces of the objects shown in Fig. 115 have been obtained by the methods just explained for Fig. 114, but none of the construction work has been shown. In both figures the H projections of the lines of intersection fall in the H projection of the prism, as indicated by the heavy-line portions of that projection.

95. The general method of finding the line of intersection of surfaces

In Fig. 116 the points M, N, O, P, and Q on the lines of intersection of the two prisms have been obtained by the piercing-point method used in Figs. 112 to 115, inclusive. The cutting-plane method has been used to find the points G, J, K, and L. The principle employed in obtaining these latter points will serve to determine the intersection of any two surfaces, plane or curved, however involved their shape may be and however disadvantageously they may be located in space. This general principle, limited to cases ordinarily dealt with, may best be expressed in rule form as follows:

> Pass a series of auxiliary cutting planes so that each one will cut from both surfaces lines that will project as straight lines or circles on the coördinate planes. The points in which these sets of lines in a single cutting plane intersect each other are points on the required line or lines of intersection of the two given surfaces. A sufficiently large number of points having been obtained, by means of several well-chosen cutting planes, they are joined together in the proper order to determine the entire line of intersection.

NOTE. The cutting planes may or may not be parallel to each other. They may or may not be projecting planes; their direction will be determined by the conditions of each problem.

In the figure the cutting planes S and T are chosen to illustrate the principle. Plane S is an H projecting plane passed through the edge line CD of the square prism. It is represented by an H trace only. Plane S cuts the two straight lines AA and BB from the triangular prism. The lines AA and BB intersect the line CD at the points K and L respectively. These are points on the lines of intersection which may be joined in the proper order to the points N, O, and Q, already determined. Similarly, the plane T, passed through the edge line EF of the square prism, cuts two lines from the triangular prism which intersect EF at the points G and J. These are also points on the lines of intersection and are joined to the points M, P, and Q to complete the intersection of the two surfaces. The nine points are joined together by means of full and dash lines to indicate the visible and invisible portions of the lines of intersection. The H projection of the square prism contains the H projections of the lines of intersection, as indicated by the heavy-line portions of that projection.

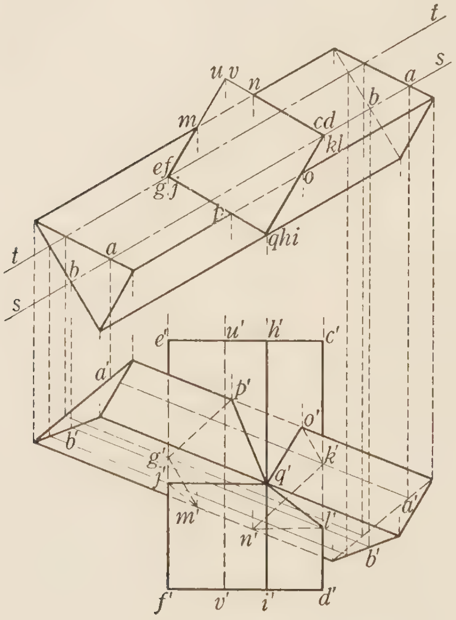


FIG. 116. Intersection of plane surfaces

96. Intersection of a plane and a curved surface or of two curved surfaces, and the determination of the invisible portions of lines drawn on a surface

The line of intersection of a plane and a curved surface may be a straight line, or it may be any plane curve, such as a circle, an ellipse, a parabola, or an irregular curve which cannot have any descriptive name given to it. Two curved surfaces may intersect in a similar manner or in a three-dimensional curve. In many instances the shape of the line of intersection can be determined by inspection and its projections drawn without making use of the general principle of determining intersections. Where possible this should be done, as shown in Figs. 117, 118, 119, and 120. In other cases the curved surface may be of such a shape, or the angles of tilt of the plane or the location of the two surfaces relative to each other may be such, that it is necessary to pass a number of cutting planes in each case in accordance with the requirements of the general intersection method of Article 95, and to determine the lines of intersection point by point. The typical cases discussed in the following articles are of such a character.

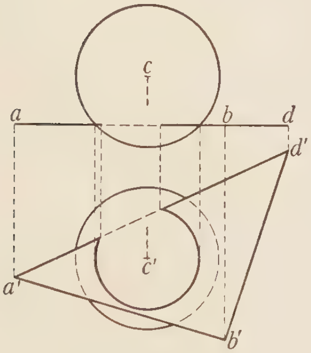


FIG. 117. Intersection of triangle and sphere

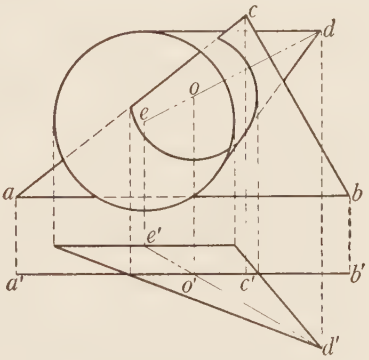


FIG. 118. Intersection of triangle and cone

The general principle or method of intersections, stated in Article 95, requires a series of similar and sequential steps in the solution of any intersection problem. It is the general analysis for all problems. Therefore a statement of how these steps are to be carried out constitutes the principal item of additional analysis needed for the solution of any problem in the intersection of surfaces by this method. Because of the universal applicability of this general principle, only such supplementary directions as are essential to make the principle easily operative in each case are included under the formal heading of Analysis in the following problems. These supplementary directions apply principally to the matter of determining the character and location of the cutting planes to be passed in each particular case.

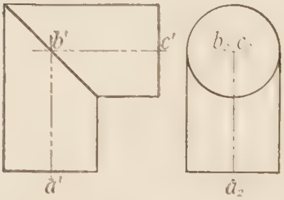


FIG. 119. Intersection of circular cylinders

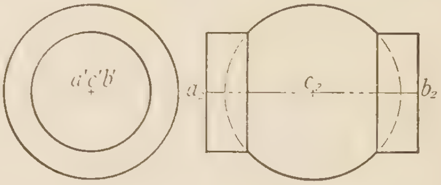


FIG. 120. Intersection of sphere and cylinder

Since a surface is considered opaque, lines that are drawn on certain portions of it are concealed from the observer by other portions of the surface. These invisible or hidden lines are usually parts of some line of intersection of the surface with a plane or other surface. They must be represented by the draftsman's broken or dash-line convention. Lines on a surface that are invisible from the point of view of the observer when he is making the elevation view of the surface may be entirely visible in the plan view. The determination of the invisible portions of a line on a surface is best made by choosing a point on the line whose projections are known and applying to it a general scheme of space analysis that will fix not only the visibility of the single point in each view, but also that of adjoining points and thus the visibility of the whole line. Several simple examples are discussed below.

Sphere. Assume that the point A on the line of intersection of the sphere with a plane has been found and its visibility is to be determined in each view, Fig. 121. Obviously this point, as well as all other points on the sphere whose projections are below the trace of the mid-section plane S, in the elevation view, is on the bottom half of the sphere. The point A is invisible from a point of view above the sphere, the direction from which the observer must look at the surface when making the plan view. Its projection in the plan view, therefore, must be on the dash-line portion of the curve. Points immediately adjoining the point A on the given line of intersection must also have their H projections on the broken-line portion of the curve. Similarly, the point A, as well as all other points on the sphere whose projections are below the trace of the mid-section plane T in the plan view, must be on the front side of the sphere and therefore visible in the elevation view. The mid-section planes, S and T, cut the circles from the sphere that are projected as the representations of the surface on the H and V coördinate planes. Since the plane S separates the visible portions of the sphere from the invisible portions when the observer is looking at the sphere from the top, and since the plane T separates the visible from the invisible portions when the observer is in front of the sphere, the points, on either

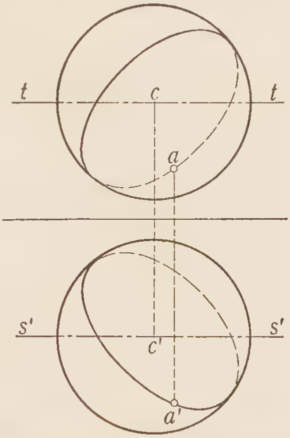


Fig. 121. Visible and invisible points on a sphere

projection of the line of intersection, at which the curve becomes tangent to the projection of the corresponding mid-circle are the dividing points between the visible and invisible portions of these curved projections. Consequently the determination of the visibility of a single point on each projection of the curve of intersection is sufficient to determine the dash-line portion of the projection as the curve is drawn through the plotted points.

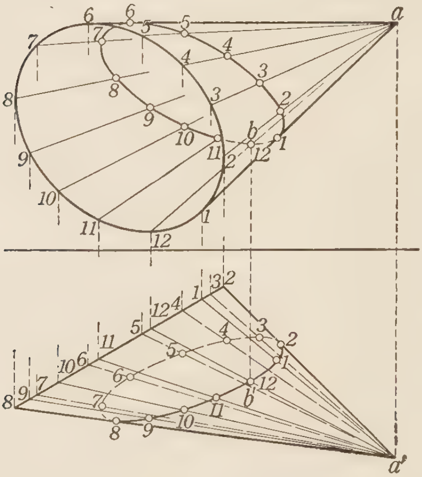


FIG. 122. Invisible lines on a cone

The foregoing discussion may be applied to other surfaces of revolution with slight modification of detail in certain instances.

Cone. Fig. 122 represents an oblique cone A, on which its line of intersection with a plane is to be traced. Assume that the point B on the line of intersection has been located and its visibility is to be determined. It is on element A12 of the cone. Obviously, from the plan view, element A12 is on the front portion of the cone; therefore it can be seen by the observer when standing in front of the cone to obtain the elevation view. Consequently the vertical projection of the point B must be on the full-line portion of the vertical projection of the curve of intersection. Points on either side of b' in this projection must be visible, and therefore a full line is drawn through them and extended until the two points of tangency with the contour elements, A2 and A8, are reached, when the curve becomes invisible and is represented by a dash line.

The element A12 is shown by the plan view of the base to be on the lower portion of the cone, as are also elements A7 to A11, inclusive. Except for the segment which is visible through the open base, it is invisible from above, the position of the observer when the plan view is being made; hence the projection of the point B, which is on element A12, must, in the plan view, be on a dash-line portion of the curve of intersection. The points of tangency of the projection of the curve of intersection with the contour elements in this view separate the full from the dash-line portions, just as in the vertical projection. Since the base of the cone is considered open, the problem of determining the visibility of the curve of intersection is further affected by the fact that any line on a surface is treated as if it were a joint or seam in the surface and therefore visible from either side. This is the reason why the portion of the projected curve in the plan view, running through points 7, 8, 9, 10, and 11, is shown as a visible line.

The discussion of the determination of the visible and invisible lines on the cone applies equally well to the cylinder. If two closed surfaces such as the cone and cylinder, for instance, intersect, there are likely to be four points of tangency of the projected curve with contour elements; hence a point on each section of the curve must be tested as outlined above to fully determine the visibility. Furthermore, each section of the line must be considered first as a line on the cone, then as a line on the cylinder, before the final judgment can be made. If either surface hides the section in question from the observer, then it must be shown as a dash line. The visibility of a point on the line of intersection of two surfaces may be decided in any projection by the following rule: If the two elements which determine a point are visible, their intersection is a visible point; if either of the elements is invisible, the point of intersection is invisible.

An important reverse application of the foregoing reasoning for determining visibility occurs in the problem of passing a plane through the base of any given cone or cylinder, when the visibility of the projections is already indicated on the drawing, as shown in Fig. 123. Here two lines must be drawn between points that are selected on the base curve, in order to determine the plane of the base. If one projection of a point is assumed, say the H projection, *a*, in the figure, and a projecting line is dropped to the vertical projection, then the question of which of two possible positions on this projection of the curve is the correct one for the assumed point can only be answered by a study of the base curve from the point of view of its visible and invisible parts. From the H projection it is seen that the point A must be on the back and upper portion of the cylinder. It is also seen to be behind portions of the cylinder that extend beyond it toward the left. Therefore point A must be invisible when the cylinder is viewed from the front, and consequently its vertical projection must be found on the dash-line portion of the vertical projection of the curve, that is, at *a′*.

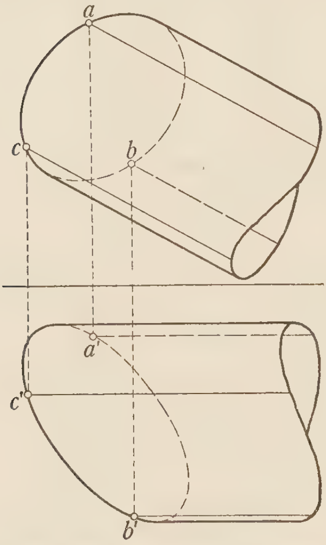


FIG. 123. Visible and invisible points on base of cylinder

By the selection of the H projection of the point B as shown in the figure and the use of a similar line of reasoning to that used in the case of the point A, it is determined that the vertical projection of B must be on a full-line portion of the vertical projection of the curve, hence at *b′*. In the case of the point C, where the H projection, *c*, is assumed and where it is determined that its vertical projection must be on a full-line portion of the vertical projection of the curve, there arises the problem of properly fixing its location, since the projector from *c* can terminate in two possible points and still satisfy the requirement. This difficulty may be overcome by considering the points immediately to the left of the point B on the base curve and noting, as they are projected one after another, that for point C to be in the sequential position indicated by its H projection, *c′* must be on the lower sweep of the vertical projection.

Lines drawn between the points A, B, and C determine the plane of the base of the given cylinder.

97. To find the line of intersection of a given cone and plane

Analysis. See the general principle of Article 95. The necessary auxiliary cutting planes should be so passed through the two surfaces that they will cut from the cone either straight lines (elements) or curves that will project in one view as straight lines and in another as circles. In the first instance the planes will all contain the apex of the cone; in the second instance none of the planes will contain the apex, but they will, in general, be parallel to the plane of the base of the cone. In each case straight lines will be cut from the given plane.

Construction One. Given the oblique elliptical cone A and the intersecting plane T, Fig. 124. Pass a series of H projecting planes, M, N, P, Q, etc., through the elements of the cone A1 to A12, inclusive. Plane P is chosen to exemplify the procedure to be followed in each case. It cuts elements A3 and A11 from the cone. It also cuts the line VH from the given plane T. The line VH intersects element A3 at the point B, a point on the required line of intersection. The line VH intersects element A11 at the point C, a second point on the required line of intersection. Other points on the line of intersection are obtained in the same way as were the points B and C, but the constructions have been omitted from the drawing. The smooth-curved lines, joining the respective H and V projections of the points thus found, constitute the two projections of the required line of intersection.

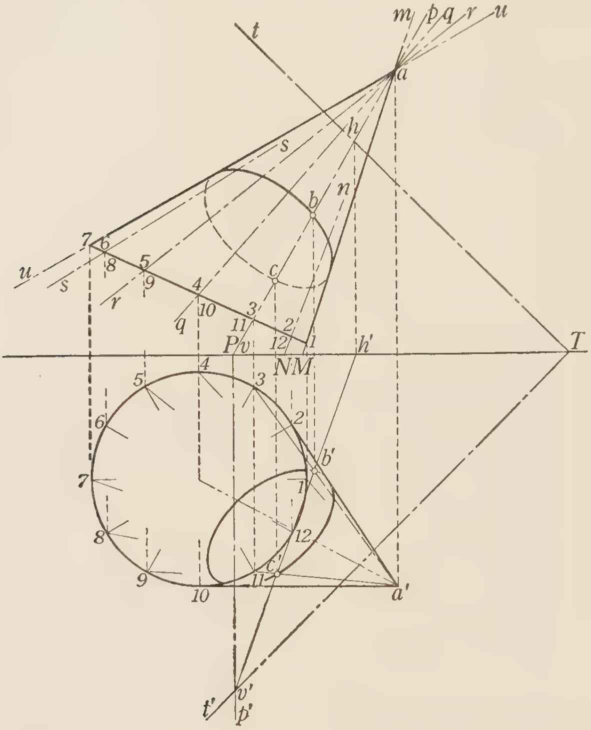


FIG. 124. Intersection of plane and cone (cutting planes through elements)

Construction Two. Given the oblique elliptical cone A and the intersecting plane T, Fig. 125. Obtain the line of centers, AC, of the cone (see Article 87 and Fig. 105). Pass a series of H projecting planes, M, N, P, Q, etc., through the cone parallel to the plane of its base. Each of these planes cuts an ellipse from the cone. Each ellipse projects upon the V plane as a circle with its center at the vertical projection of the point in which the line of centers, AC, pierces the plane cutting the ellipse from the cone. The diameter of each circle is such as to make the circle tangent to the vertical projections of the two contour elements of the cone. Each ellipse projects upon the H plane as a straight line coinciding with the H trace of the cutting plane. For example, the plane N cuts an ellipse from the cone which projects upon the V plane as a circle with its center at c_1', and

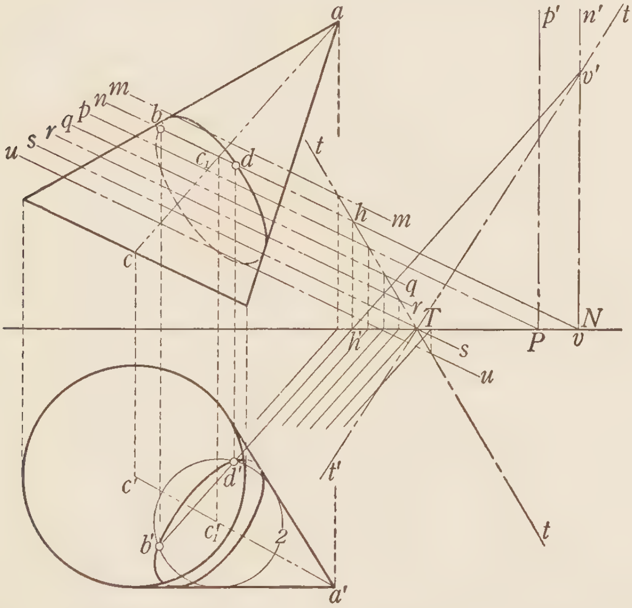


FIG. 125. Intersection of plane and cone (cutting planes parallel to base)

upon the H plane as a straight line coinciding with the H trace of N. The plane N cuts the line VH from the plane T. The line VH intersects the ellipse whose center is C_1 at the points B and D, which are on the required line of intersection. Other points on the line of intersection are obtained in the same way as were the points B and D, by the use of the planes M, P, Q, R, etc. The smooth-curved lines joining the projections of these points constitute the projections of the required line of intersection.

Discussion. This method of finding the line of intersection of a cone and a plane is not always a satisfactory one because of the confused condition of the drawing where the circular projections converge. However, since the lines cut from the given plane by the series of projecting planes are parallel to each other, the method is a very rapid one to use. Only the projections of one piercing point of each line of intersection need be found.

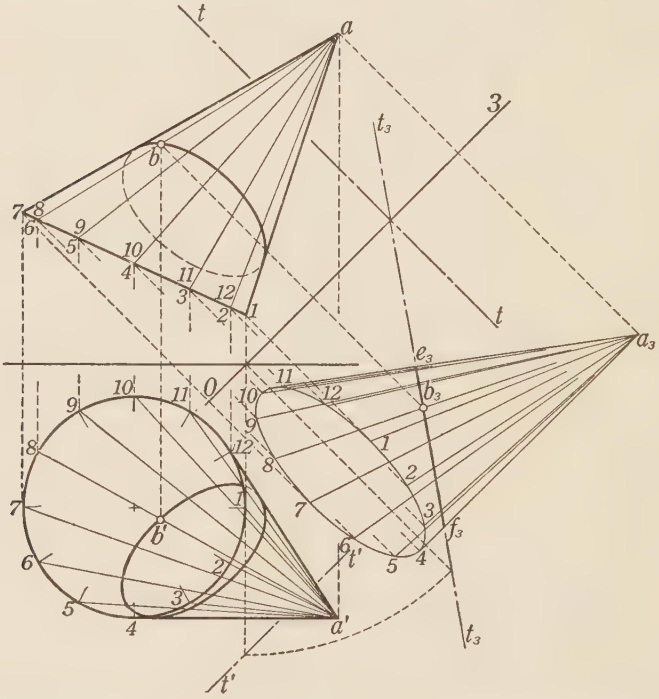


FIG. 126. Intersection of plane and cone (auxiliary-plane method)

Construction Three. Given the oblique elliptical cone A and the intersecting plane T, Fig. 126. Set up the auxiliary coördinate plane 3 perpendicular to the H plane and to the plane T and project the cone A upon it. This is best done by first projecting the apex of the cone to a_3 and then projecting a sufficient number of base points, 1, 2, 3, 4, etc., upon plane 3 to determine a smooth curve for this new base projection. The base points should be selected carefully with a view to keeping the constructions as open as possible and at the same time furnishing enough points to insure accurate intersections. Straight lines joining each base-point projection to a_3 constitute the plane-3 projections of a selected group of elements of the cone. Other elements may be inserted at desirable places. Plane T shows as the trace t_3t_3 on plane 3. This trace is the edgewise view of the plane T. The full-line segment, e_3f_3, of t_3t_3 is the edgewise view of the line of intersection of the plane T and the cone A. The H and V projections of points on the required line of intersection are obtained from this auxiliary view by returning the point of intersection of each element, A1 to A12, with e_3f_3 to the H and V projections of the corresponding elements by means of the usual constructions. For example, b_3 is one of these points at the intersection of $a_3 8$ and e_3f_3. It returns to b on the H projection of element A8 and to b' on the vertical projection of A8, as shown in the figure. The H and V projections of the points of intersection of the other eleven elements are found in a similar manner, and smooth curves are drawn through each set of projections thus determined.

Construction Four. Given the oblique elliptical cone A and the intersecting parallelogram BCDE, Fig. 127. Set up the auxiliary coördinate plane 3 perpendicular to the plane of the parallelogram by drawing *0–3* perpendicular to *cd*, the H projection of a line parallel to the H plane and lying in the plane of BCDE. Project the cone A upon this auxiliary plane in the manner explained in Construction Three and shown in Fig. 126. Project the parallelogram BCDE upon plane 3 at $b_3c_3d_3e_3$. This straight-line projection is the edgewise view of the parallelogram and also of its intersection with the cone. The plane-3 projections of a number of points on the required line of intersection are located by marking the points of crossing of the plane-3 projections of the twelve selected elements, A1 to A12, and the line $b_3c_3d_3e_3$. The H and V projections of these points are obtained by the usual return constructions, and a smooth curve is then drawn through the projections of each group.

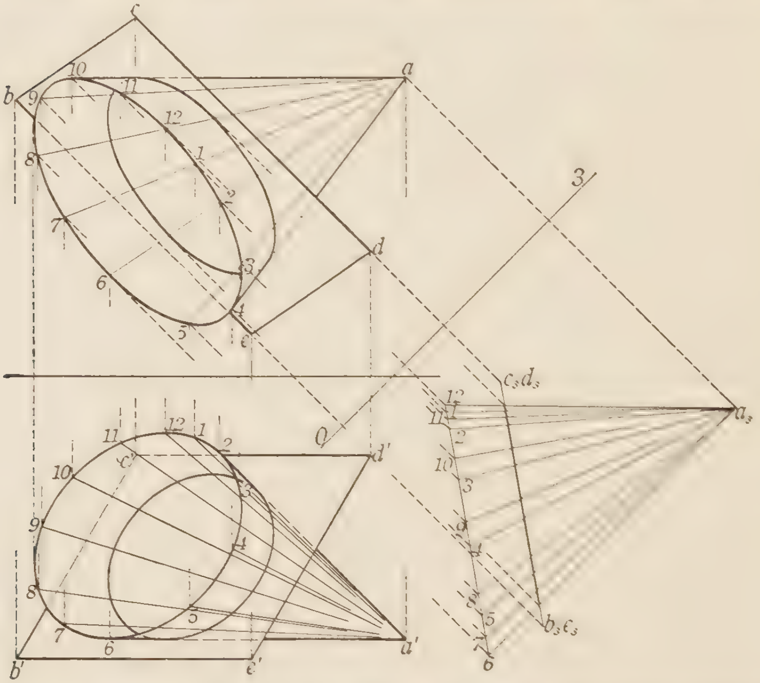


FIG. 127. Intersection of cone and parallelogram (auxiliary-plane method)

Discussion. Several important details, developed in the four constructions of this article, should be emphasized:

1. Projecting planes should be used as the cutting planes where possible, and they should be so spaced that a uniform distribution of points on the line of intersection will result (Fig. 124).
2. Numerical markings on the projections of the bases of cones and other surfaces are more satisfactory in inter-

section problems than literal markings, in that the range is unlimited and no duplication of other important lettering on the figures can occur. All prime and subscript designations of these numbers may be dispensed with without any confusion resulting. These numerical markings should be put on each coördinate view of the base, as shown in Figs. 126 and 127, and even on the projections of the line of intersection in complicated cases.

3. The base numbering may well be done at the outset of the construction and in such a way that a single line projection in one view may suffice for two elements on the cone, as, for example, the H projections of the elements in Fig. 126.
4. Extra planes should be inserted when it is found that the series of planes chosen leaves too wide a gap between certain of the determined points on any projection of the required line of intersection.

Since a cylinder is simply a cone with its apex at infinity, the foregoing analysis and constructions may be used to determine the line of intersection of a plane and a cylinder.

98. To find the line of intersection of a given surface of revolution and a plane

Analysis. See the general principle of Article 95. The auxiliary cutting planes must be passed perpendicular to the axis of revolution of the given surface, thereby cutting circles from that surface. Straight lines will be cut from the given plane.

Construction One. Given the prolate ellipsoid of revolution and the intersecting plane R, Fig. 128. The axis of revolution of the ellipsoid is the line AB perpendicular to the H plane. Pass the V projecting planes M, P, Q, etc. through the ellipsoid and perpendicular to AB. Plane P is chosen to exemplify the procedure to be followed with each plane. It cuts the straight line VE from the plane R and a circle from the ellipsoid. The vertical projection of the circle is the straight line $c'd'$ in the V trace of plane P; its horizontal projection is the circle *cod* with its center at a. The line VE intersects the circle COD at the points N and O on the required line of intersection. Other points on the line of intersection are obtained in the same way by using other planes of the series, but the constructions have been omitted from the drawing. The smooth-curved lines joining the respective projections of the points thus found constitute the projections of the required line of intersection.

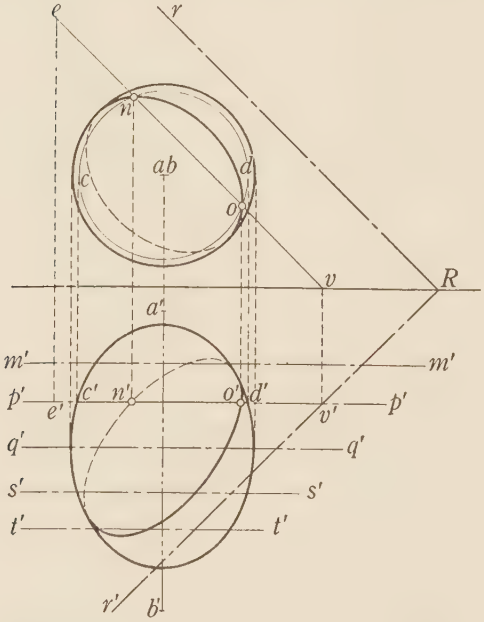


FIG. 128. Intersection of plane and ellipsoid of revolution

Construction Two. Given the prolate ellipsoid of revolution and the intersecting plane S, Fig. 129. Set up the auxiliary coördinate plane 3 perpendicular to the plane S and parallel to the axis, AB, of the ellipsoid. Project the ellipsoid upon plane 3. It appears as an ellipse identical in size with that of the vertical projection, since both plane 3 and the V plane are parallel to the axis, AB. The trace s_3s_3 is the edgewise view of the plane S and contains c_3d_3, the plane-3 projection of the required line of intersection. Pass the auxiliary plane-3 projecting planes M, P, Q, etc. through the ellipsoid and perpendicular to AB. Plane P is chosen to exemplify the procedure to be followed with each plane. It cuts a circle from the ellipsoid which projects as the straight line e_3f_3 on plane 3, as the circle *egfk* on the H plane, and as the straight line $g'k'$ on the V plane. The plane-3 projection of the line cut from the plane S by the plane P is a point and coincides with the point of intersection of e_3f_3 with c_3d_3. Consequently this point is the plane-3 projection, n_3o_3, of two points, N and O, on the required line of intersection. These points are returned in the usual manner to the H and V projections of the circle EGFK, as *n* and *o*, and *n′* and *o′*, respectively. Other points on the line of intersection are obtained in the same way as were points N and O. The smooth-curved lines drawn through the H and V projections of the points thus found, together with c_3d_3, constitute the three projections of the required line of intersection.

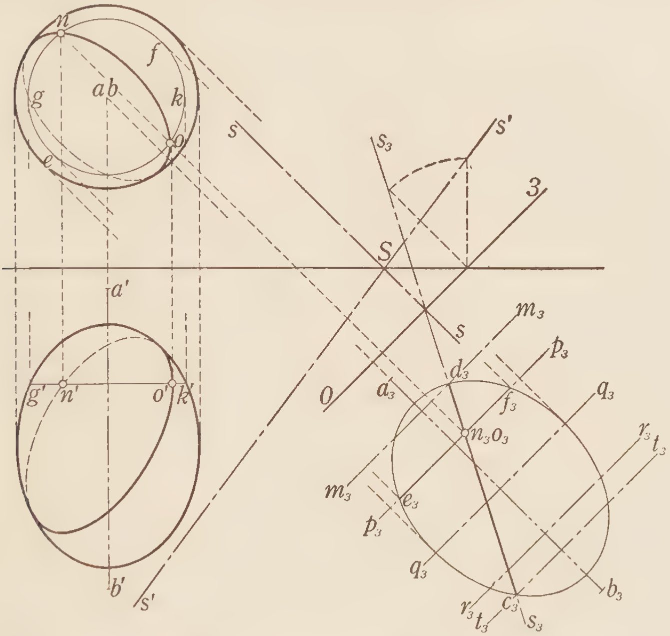


FIG. 129. Intersection of plane and ellipsoid of revolution

Discussion. Except when the axis of revolution is inclined to two or all the principal coördinate planes, the auxiliary-plane method has little advantage over the method of Construction One in solving problems of intersection where the sphere or ellipsoid of revolution is involved. One point of importance in favor of the method lies in the fact that the position of the line of intersection on the surface of revolution can be told readily from the auxiliary view; hence the cutting planes can be spaced more effectively and within definite limits on the drawing. In no case should an auxiliary plane be used in such problems unless set up parallel or perpendicular to an axis of revolution.

99. To find the line of intersection of two given cones

Analysis. See the general principle of Article 95. The auxiliary cutting planes must be passed through the two cones so that they contain the line joining the apexes of the cones, thereby cutting elements from each cone; or, in certain instances, they may be passed parallel to the bases of the cones if, by so doing, they will cut sections from the cones that project as circles on one of the coördinate planes and as straight lines on another.

Construction. Given the two intersecting oblique elliptical cones A and B, Fig. 130. Connect the apex of the cone A to the apex of the cone B by the straight line AB. Pass the series of cutting planes M, N, Q, R, S, and T through the line AB. Each plane, except the two tangent planes M and T, cuts two elements from each cone. Each element cut from the one cone by a cutting plane intersects the two elements cut from the second cone by the same cutting plane in two points that are on the required line of intersection. For example, plane R cuts the two elements numbered A4 and A4 from cone A and the two elements numbered B4 and B4 from cone B. The determination of the four elements which plane R cuts from the two cones is made easy in this particular case by the fact that each base is in one or another of the coördinate planes. Element A4 on the upper portion of cone A intersects the two elements

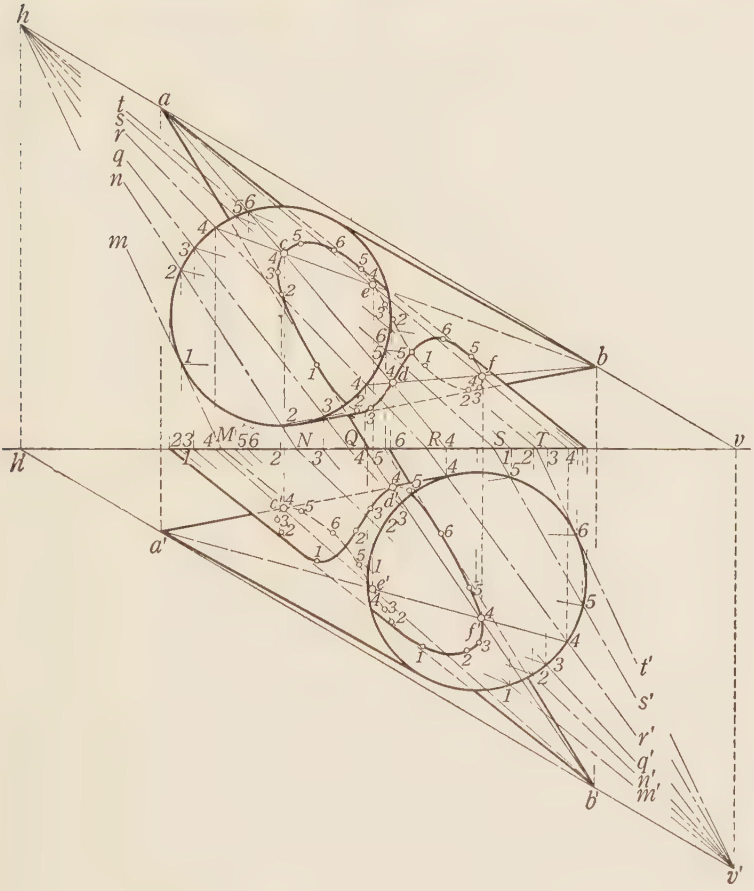


FIG. 130. Intersection of two oblique cones (cutting planes through elements)

B4 and B4, on the cone B, at the points C and D. Element A4 on the lower back portion of cone A intersects elements B4 and B4 at the points E and F. Other points on the required line of intersection are obtained in a similar manner, but the constructions have been omitted from the drawing. Smooth curves are drawn through the respective projections and the invisible portions represented in each view by the usual dash-line convention.

Discussion. If the base of either of the two intersecting cones is not in a coördinate plane, it will be necessary to produce that cone until its base is in one or another of the coördinate planes. If both bases are in one coördinate plane, only one trace of each cutting plane need be drawn; otherwise, two traces are needed. If the bases of the two cones are in parallel projecting planes and show on one coördinate plane as circles, then the cutting planes may be passed parallel to these base planes and the constructions carried out as in Figs. 132 and 135 (see Construction Two of Articles 100 and 101 for an explanation of the method suggested).

In the problem of finding the intersection of a plane and cone (Figs. 124 and 127) it was found sufficient in the constructions to mark the elements of the cone in a simple numerical order without reference to the cutting planes. In more complicated cases, like that of Fig. 130, it is essential that a somewhat elaborate system of numbering be employed and one that definitely ties up with the cutting planes. Probably the most practical system is that used in Fig. 130. Plane M is considered to be the first cutting plane passed. The elements cut from both cones by this plane are numbered 1 at their intersection with the bases of the cones. Plane N is considered the second plane passed; elements cut from the cones by it are numbered 2 at the base points. Other elements are numbered in order as successive planes are passed. If an insert plane is needed, say, for example, between planes N and Q, then the elements cut from the cone are numbered 2A; if the insert plane is passed between planes Q and R, the elements are numbered 3A; and so on for any number of insert planes.

In order to connect the projections of the determined points of intersection in the proper order it is often desirable to extend the numbering scheme to the projections themselves. The H and V projections of the line of intersection of the two cones of Fig. 130 are treated in that way. On them is given the point numbering that results from the employment of the system described in the preceding paragraph. A careful study of the drawings will serve to develop the advantages of the numbering system recommended and the principle upon which it is based.

100. To find the line of intersection of a given cone and cylinder

Analysis. See the general principle of Article 95. Since a cylinder is the special case of a cone with its apex at infinity, the Analysis of Article 99 is applicable to this problem. The line which must be drawn through the apex of the cone to control the direction of the cutting planes will necessarily be parallel to the axis or elements of the cylinder. In special cases where the base of either or both surfaces projects as a circle upon one of the coördinate planes and as a straight line upon either of the other two, the cutting planes may be passed through the elements of the cylinder parallel to the base of the cone, or they may be passed parallel to both bases of the surfaces, depending upon the given conditions.

Construction One. Given the oblique elliptical cone A and the intersecting oblique elliptical cylinder B, Fig. 131. Since the base of the cone A, as given, does not lie in a coördinate plane, the cone is extended until it intersects the V plane. The constructions for this part of the solution of the problem are shown in light lines on the drawing. The elements used in extending the cone are not the same as those cut from it by the planes M to T, hence the necessity for the double system of numbering shown. Draw the line AH through the apex of the cone and parallel to the elements of the cylinder. Through the line AH pass the series of cutting planes M, N, Q, R, S, and T. Except M and T, each plane cuts two elements from the cone, which intersect each of the two elements cut from the cylinder in points on the required line of intersection. For example, plane S cuts the elements numbered 5 from each surface. Element A5 on the upper front portion of the cone intersects the two elements numbered 5 on the cylinder at points C and D. Element A5 on the lower front portion of the cone intersects the two elements numbered 5 on the cylinder at points E and F. Other points on the line of intersection are obtained in a similar manner and smooth curves drawn through their

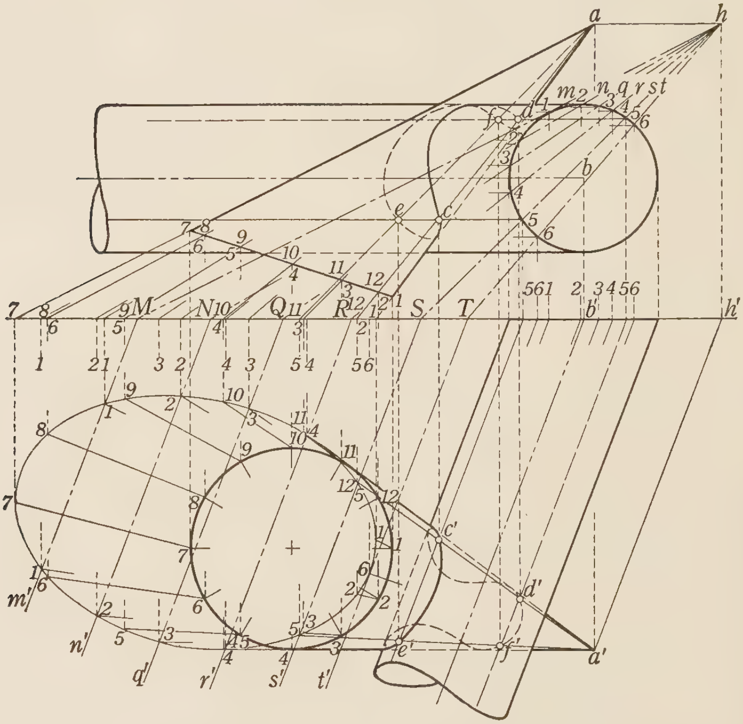


FIG. 131. Intersection of oblique cone and cylinder (cutting planes through elements)

projections with the usual dash-line convention showing the invisible portions in each view.

Construction Two. Given the oblique elliptical cone A and the intersecting oblique elliptical cylinder B, with their bases lying in parallel H projecting planes and projecting as circles on the V plane, Fig. 132. Pass the series of cutting planes M, N, Q, etc. parallel to the planes of the two bases of the given surfaces. Each plane cuts ellipses from the surfaces, which project as circles on the V plane and as straight lines on the H plane. The intersections of each pair of circles are the V projections of two points on the required line of intersection. For example, plane N cuts ellipses from the cone and cylinder that are represented on the V plane by the two circles whose centers are c' and d' respectively. The H projections of the ellipses lie in the H trace of plane N. The two circles intersect in the points e' and f', from which e and f are projected to establish the two points E and F. Other points on the curve of intersection are obtained in a similar manner and smooth curves drawn through their corresponding projections. Since the curves cut from the surfaces by the plane M do not intersect, an insert plane N_a not shown on the drawing, is used to obtain points on the line of intersection between planes M and N.

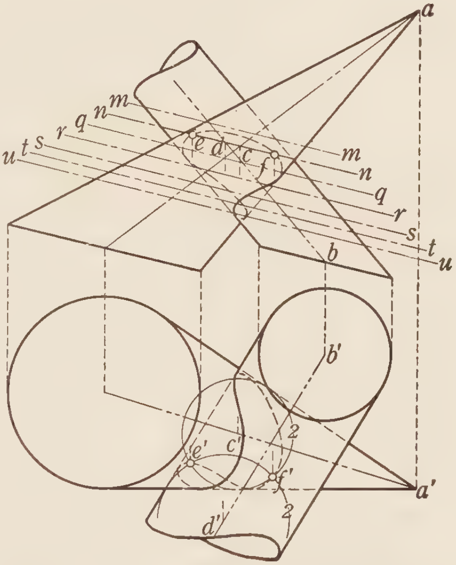


FIG. 132. Intersection of cone and cylinder (cutting planes parallel to bases)

The numbering system developed in Article 99 may be applied to advantage in this construction to keep the circular projections properly correlated with each other and with the cutting planes.

Discussion. Although the method explained in Construction Two permits of the very rapid determination of points on the line of intersection and should be used when conditions are favorable to its application, in many instances the numerous overlapping of the circles in one view causes confusion and inaccuracy of construction and may necessitate the abandonment of the method in favor of that employed in Construction One.

Construction Three. Given the oblique elliptical cone A and the cylinder B with its elements parallel to the plane of the base of the cone, Fig. 133. Pass a series of cutting planes M, N, P, Q, etc. through the elements of the cylinder and parallel to the plane of the base of the cone. Each plane, except the two tangent planes M and Z cuts two elements from the cylinder and an ellipse from the cone that show, respectively, as straight lines and a circle in H projection. The points of intersection of the circle with the projection of one or both of the elements cut from the cylinder by a single plane give H projections of one, two, three, or four points on the required line of intersection, from which the V projections are readily found. For example, plane Q cuts the two elements from the cylinder that are numbered 4 in the figure. It cuts an ellipse from the cone that projects upon the H plane as circle 4 with its center at *c*. The H projections of the intersections of one element numbered 4 with the circle 4 give the points *e* and *f*, from which *e*′ and *f*′ are projected in the usual way. The second element cut from the cylinder by the plane Q does not intersect the cone, since circle 4 does not intersect the H projection of this element. Other points on the line of intersection are obtained in a similar manner and their projections connected by smooth curves. Element 1 on the cylinder does not pierce the cone, but this fact could not be foreseen at the beginning of the construction, when the cutting plane M was passed.

Discussion. It should be noted that the method of Construction Two is identical in principle with that of Construction Three and does not require that the bases of the surfaces

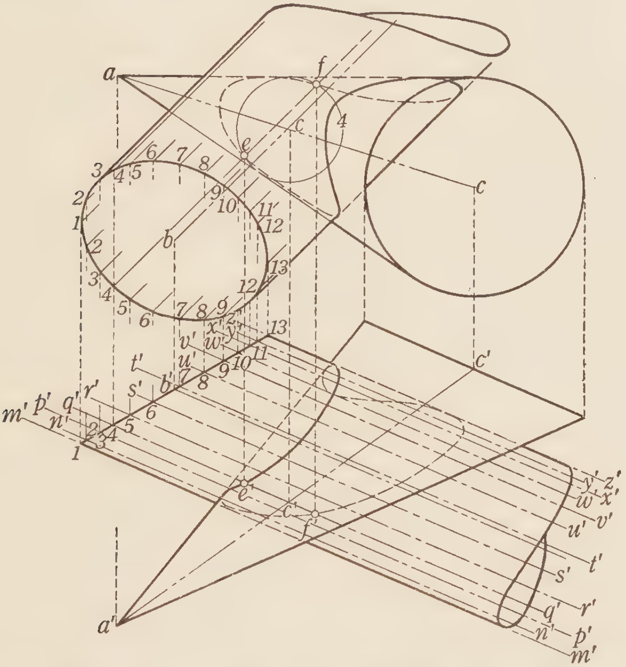


FIG. 133. Intersection of cone and cylinder (cutting planes through elements of cylinder and parallel to base of cone)

be in the coördinate planes. The chief advantages of the method are that all the constructions are kept within the limits of the drawing of the problem itself; that no ground lines are needed; that the surfaces do not have to be extended to obtain bases in the coördinate planes; and that in every way the work of obtaining the line of intersection is shortened and made simple. The method cannot be applied to every set-up, however, with an equal degree of success. Only the judgment of the draftsman can be relied upon as a guide in the choice of method in any given problem.

101. To find the line of intersection of two cylinders

Analysis. See the general principle of Article 95. The cutting planes to be passed through the two cylinders must all be parallel to a direction plane that contains two lines drawn through any selected point in space and parallel, respectively, to the elements of the cylinders. The cutting planes may, however, be passed parallel to the plane of the base of either or both cylinders if the sections cut from the cylinders project upon one of the coördinate planes as circles and upon another as straight lines.

Construction One. Given the two oblique intersecting cylinders A and B, Fig. 134. Through any selected point O draw the line OH parallel to the elements of the cylinder A, and draw OV parallel to the elements of the cylinder B. Plane W, passed through OH and OV, is the directing plane for all cutting planes. Pass the series of cutting planes M, N, Q, etc. through the cylinders and parallel to the plane W. Each plane, except the tangent planes M and U, cuts two elements from each cylinder. Each element that is cut from the one cylinder by a cutting plane intersects the two elements cut from the second cylinder in two points that are on the required line of intersection. For example, plane R cuts the elements numbered 4 from the cylinders A and B. Element 4 on the lower back portion of cylinder A intersects the two elements numbered 4 of cylinder B in the points C and D. Element 4 on the upper front portion of cylinder A intersects the two elements numbered 4 of cylinder B in the points E and F. Other points on the line of intersection are obtained similarly and smooth curves drawn through their projections. It should be noted that, outside of the matter of determining the direction of the cutting planes, the procedure for finding the elements cut from the two cylinders and their points of intersection with each other is like that for the two cones of Fig. 130, Article 99, to which reference should be made for further details of construction.

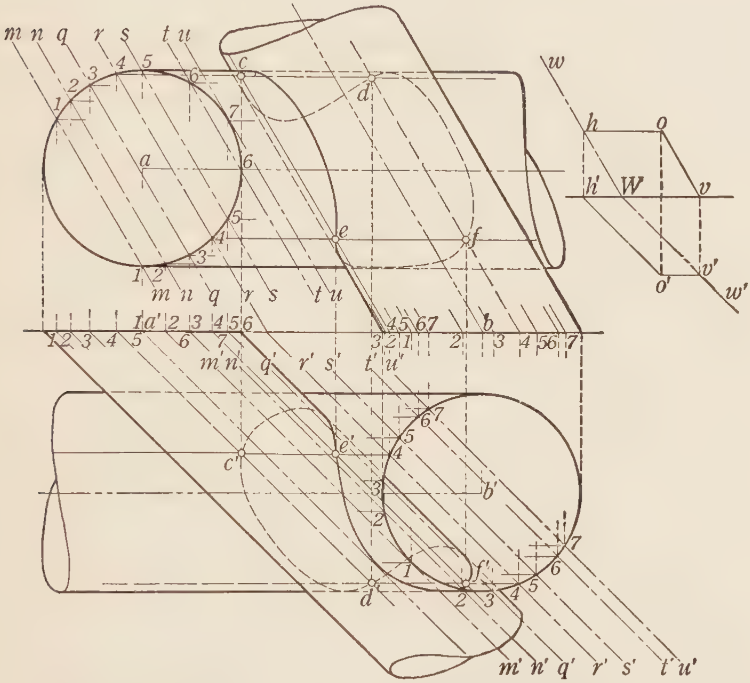


FIG. 134. Intersection of two oblique cylinders (cutting planes through elements)

Construction Two. Given the two oblique intersecting cylinders A and B, with their bases in parallel V projecting planes and projecting as circles on the H plane, Fig. 135. Pass the series of evenly-spaced projecting planes M, N, Q, R, etc. parallel to the planes of the bases of the two cylinders.

Each plane cuts an elliptical section from each cylinder, which projects as a circle upon the horizontal plane. The two ellipses cut from the cylinders by each cutting plane intersect in two points of the required line of intersection, which are indicated in the horizontal projection by the crossing of the two circles. For example, the fifth plane, S, cuts an ellipse from the cylinder A which projects upon the H plane as the circle whose center is at *c* and whose diameter is fixed by the necessary two points of tangency of the circle with the projections of the contour elements. Plane S cuts an ellipse from the cylinder B which projects as a circle upon the H plane with center at *d* and tangent to the projections of the contour elements of the cylinder B. These

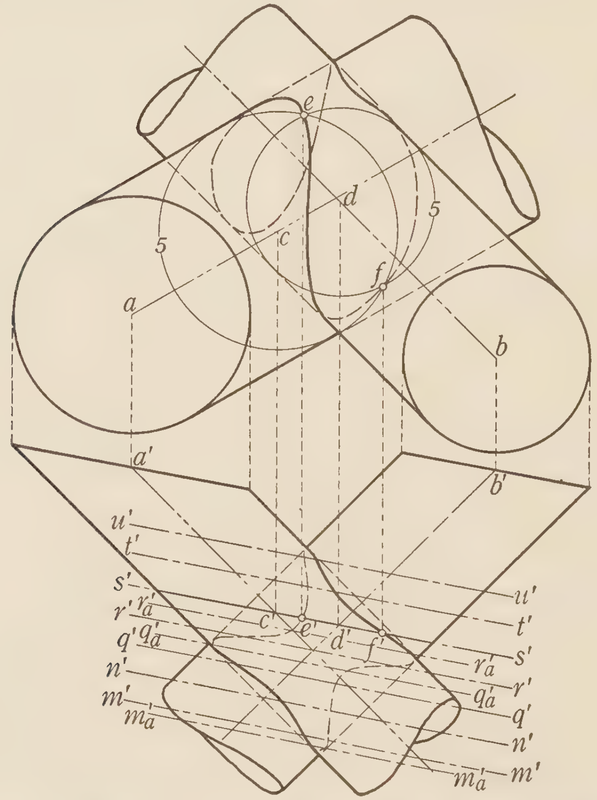


FIG. 135. Intersection of two oblique cylinders (cutting planes parallel to bases)

circles, which are numbered 5, intersect at *e* and *f*, which project to *e′* and *f′* in the vertical view. Other points on the line of intersection are obtained in a similar manner, but before

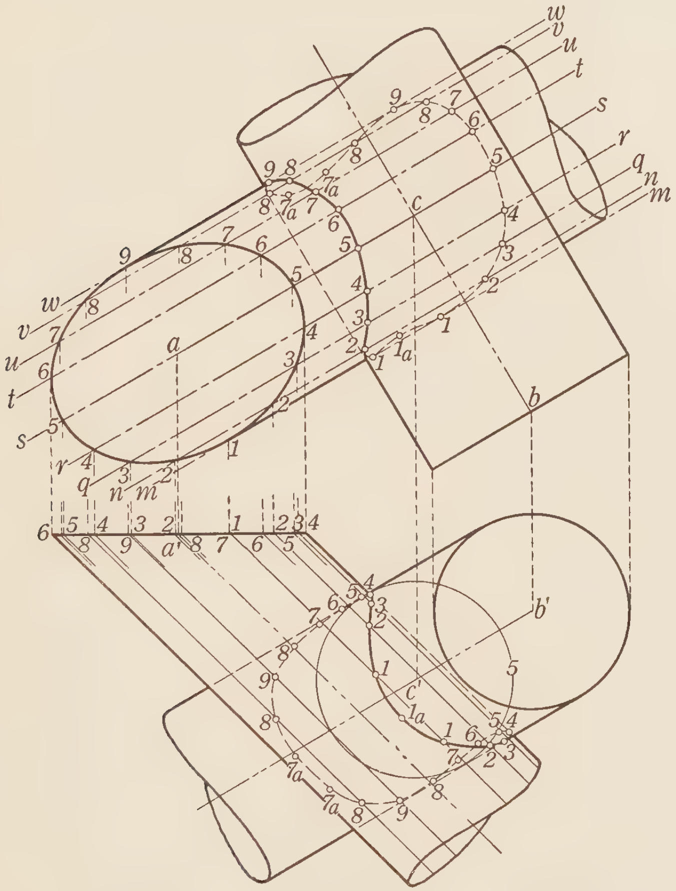


FIG. 136. Intersection of two oblique cylinders (cutting planes through elements parallel to base)

smooth curves can be drawn through their projections, insert planes M_a, Q_a, and R_a are passed to secure additional points on the line of intersection at critical places in its curvature, a procedure often carried out in intersection work.

Construction Three. Given the two oblique intersecting cylinders A and B, with the elements of cylinder A parallel to the plane of the base of cylinder B and with the base of cylinder B projecting as a circle on the V plane and as a straight line on the H plane, Fig. 136. Pass the series of cutting planes M, N, Q, R, etc. through the elements of cylinder A and parallel to the plane of the base of cylinder B. The procedure for determining the individual points on the required line of intersection is identical with that of Construction Three, Fig. 133, of Article 100, to which reference should be made for further details of construction. Attention is again called to the results of extending the base numbering system to the projections of the line of intersection. The advantages of the system are self-evident when used in this way. The use of insert planes, though not shown in the drawing, for locating points 1A and 7A should be noted.

Construction Four. Given the two oblique intersecting cylinders A and B, Fig. 137. Set up an auxiliary coördinate plane 3 perpendicular to the axis of cylinder B and project the two cylinders upon this plane. This is accomplished most readily by first projecting the axis of each cylinder upon plane 3 as a_3c_3 and as b_3d_3, as shown in the figure, then projecting each numbered point on the base of each cylinder upon plane 3. Draw smooth curves through these points. They are the plane-3 views of the bases and contain the terminal points of all elements to be drawn parallel to the axes already projected upon the plane. Cylinder B, being perpendicular to plane 3, is shown completely upon that plane by the plane curve drawn through the projections of the base points. This curve coincides with or contains the plane-3 projection of the required line of intersection of the two cylinders. The plane-3 projection of each numbered element of cylinder A is drawn parallel to a_3c_3. At the points of crossing, in the plane-3 view, of the projected elements of cylinder A and the curve representing the end view of cylinder B, projections of definite points on the required curve of intersection are located which may be returned to

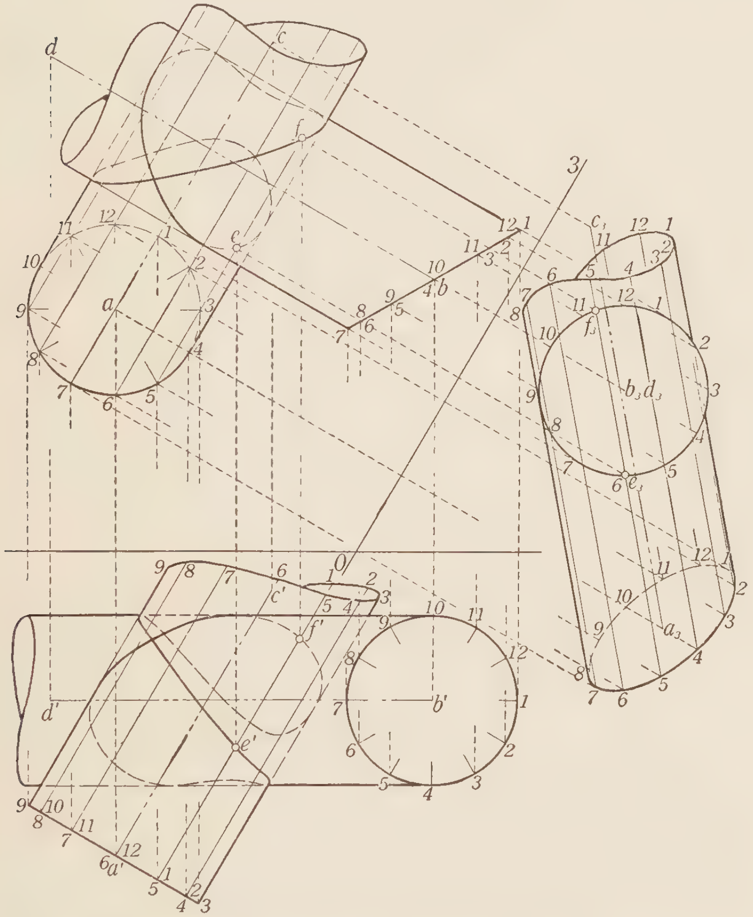


FIG. 137. Intersection of two oblique cylinders (auxiliary-plane method)

the H and V projections by the usual construction lines, as shown for the points E and F on element 5 of cylinder A.

Discussion. The end view shows that the two cylinders A and B are tangent to the same plane, A along its element 8 and B along an element between 8 and 9, and that cylinder B is circular in cross section, facts not readily discovered from the other two views. The end view also assists in determining the portions of the curve that are invisible in the other two views. The method of Construction Four is applicable to any problem involving the cylinder and does not require that a base be in any coördinate plane, as does Construction One.

102. To find the line of intersection of a given surface of revolution and a cone or cylinder

Analysis. See the general principle of Article 95. The cutting planes to be passed in this instance should all be perpendicular to the axis of the surface of revolution, thus insuring that circles will be cut from this surface, while the lines cut from the cone may be any of the conic sections, or, in the case of the circular cylinder, they may be straight lines, circles, or ellipses. The intersections of the lines that are cut from the two surfaces by any plane are individual points on the required line of intersection.

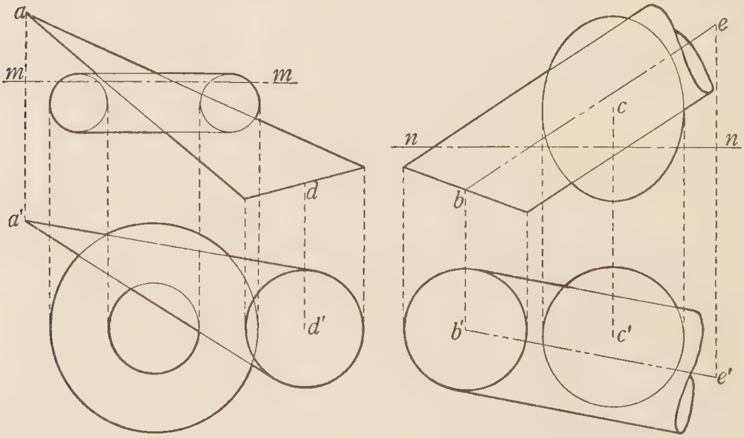


FIG. 138. Intersection of cone or cylinder with a surface of revolution

Discussion. In general cases, such as those illustrated in Fig. 138, it will be necessary to find the line of intersection of each cutting plane with the cone or cylinder by one of the constructions of the previous articles. For example, the plane M in the first figure cuts an ellipse from the cone; in the second figure the plane N cuts an ellipse from the cylinder. The projections of the ellipses on the horizontal plane are found by the methods of Article 97. Since this must be done for every cutting plane used in the solutions, the process becomes a very tedious one in most cases. But when the surface of revolution is a sphere, the construction may be shortened considerably, because of the fact that any diameter of the sphere may be treated as its axis; hence the cutting planes may be passed in the most advantageous way with respect to the cone or cylinder. The constructions for this case only are shown.

Construction. Given the sphere with its center at C, and the oblique intersecting cylinder A, Fig. 139. Since the center of the sphere is on the axis of the cylinder, and since the diameter of the sphere is larger than that of the cylinder, there will be two separate curves of intersection, on opposite sides of the sphere. Pass the series of cutting planes M, N, Q, R, etc. through predetermined elements of the cylinder. Each plane is an H projecting plane and cuts a circle from the sphere. Set up the auxiliary coördinate plane 3 parallel to the elements of the cylinder and the cutting planes. The circles cut from the sphere by planes M, N, Q, etc. project upon the auxiliary plane as circles with centers at c_3. The elements cut from the cylinder project upon plane 3 in true length and parallel to a_3c_3. They cross the corresponding circles in points which are the plane-3 projections of individual points on the required line of intersection. These projections are readily returned to H and V projections, as shown in the figure. For example, plane S cuts elements 3 and 11 from the cylinder and a circle from the sphere, which project upon plane 3 as lines *3* and *11* and the circle $b_3d_3e_3f_3$ respectively. The plane-3 projections, b_3, d_3, e_3, and f_3, where lines *3* and *11* cross the circle, determine points B, D, E, and F on the required lines of intersection. Other points on both intersections are obtained in the same way and smooth curves drawn through their projections. In determining the invisible portions of the plane-3 projections of the curves of intersection, the student should remember that the observer is supposed to be standing, in this case, to the rear of the plane and object and looking toward the front. This is in accordance with the rule that in third-quadrant projection the coördinate plane must be between the observer and the object, whether the plane is one of the principal planes or an auxiliary plane.

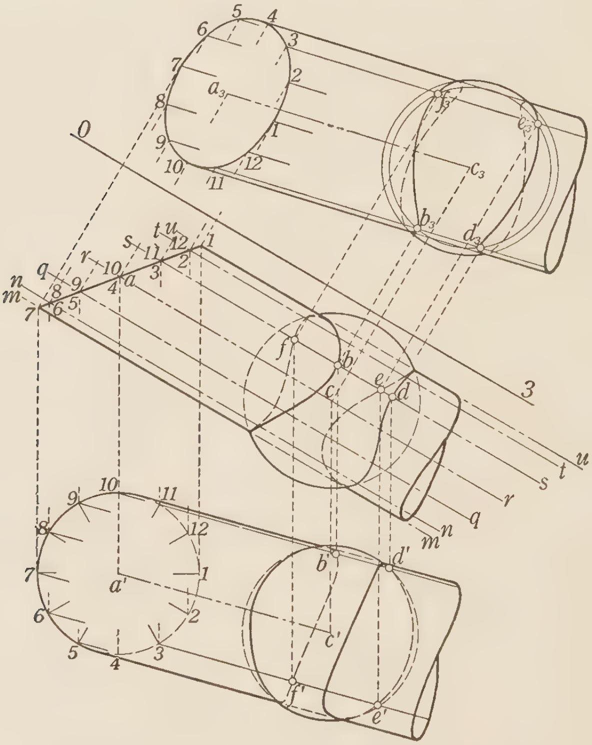


FIG. 139. Intersection of sphere and cylinder

103. Development of surfaces

To develop a surface means to roll or lay it out on a plane surface. If the surface is closed, it must be cut open along some convenient element and the development started on this element. The surfaces which may be developed are the cone, the cylinder, and the convolute. All others are undevelopable surfaces, although approximate developments may be made of many of them.

The purpose of developing a surface is to secure a pattern or template which may be used as a guide in cutting the shape from flat material, such as sheet metal, which may then be bent or rolled into the desired form and connected to other given surfaces. Developments made from scaled shop drawings must be increased to full size before they can be used as cutting patterns.

104. Sheet-metal developments

In engineering practice great numbers of sheet-metal contrivances are employed in boilers, stacks, tanks, elbows, breechings, hoods, hoppers, bins, conveyors, reducing sections, transition pieces, ventilators, etc.; in such forms as automobile bodies, hoods, fenders, and radiator elements; in fan blades, ship plates, and spiral conveyor segments; in pressed pulleys, wheels, rollers, gears, and gear guards, grease cups, and housings; in all sorts of tool parts and household utensils; in forms for concrete construction in vaults and arches; in cornice linings and architectural ornaments; and in hundreds of other forms seen in everyday use.

In the manufacture of these devices the process may vary from the simple folding of the flat sheet or plate, through more or less complex rolling, twisting, and stretching treatments, to heavy pressing operations that change the blank into the desired form.

A study of the geometrical properties of the surfaces of these manufactured devices shows them to be made up, in many cases, of planes, together with portions of cones or cylinders. If they are composed of these simple elements and the metal from which they are made is thin, they may be rolled or folded into shape with negligible stretching or compressing of the metal in the process. All other forms may be rolled from the flat only with more or less twisting or drawing of the metal as they assume their final shapes. A very important requirement in development work is the ability to analyze the shape into its component parts and to recognize these parts as being either the whole or a portion of some simple geometrical form. After this step, it will be found that the development can be derived with comparative ease, however irregular or complicated the object may be. Figs. 140 and 141 will assist the student in beginning such analyses.

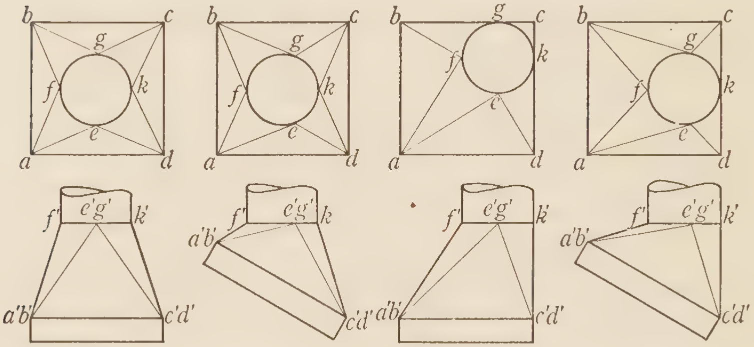


FIG. 140. Transition pieces; square to round opening (reduction hoods)

The four transition sections of Fig. 140 are made up of parts of four oblique cones and four plane triangles. The transition is from a square to a round opening. In Fig. 141 the transition is from an oblong opening with circular ends to a round opening. Each section is made up of two plane triangles and parts of two cylinders.

In comparatively simple work the layout man in the shops makes the development directly on the metal sheet, using a straightedge, compasses, steel square, scale, and a sharp steel or soapstone scriber. In more complicated work the draftsman or layout man must first construct a pattern of heavy paper or cardboard, called template-board, which may be rolled to shape to test the correctness of the development before it is copied in metal.

Unless the metal is very thin, an allowance must be made in the development because of the thickness of the material, especially if two or more pieces are to be joined together.

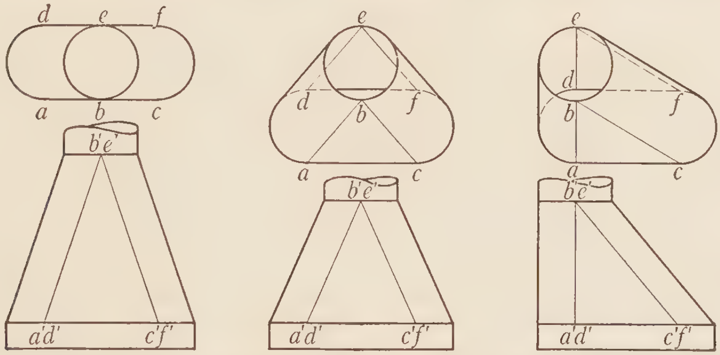


Fig. 141. Transition pieces; oblong to round opening (reduction hoods)

When rolled, a sheet is stretched on the convex side and compressed or crimped on the concave side in such manner that the central or near-central plane is unchanged. This near-central plane is called the neutral plane. The following rule may be used to make the proper allowance for thickness: For a circular drum the development will require a piece of metal whose length is the inside circumference of the cylinder increased by 3.1416 times the thickness of the metal, or the outside circumference decreased by an equal amount.

Allowance must be made for a lap at each joint if the segments are to be riveted or telescoped together. Also, it will be well to allow a small amount, best determined by experience, for inaccuracy in workmanship. This is the more necessary if the surfaces involved are only approximately developable, where the possibilities of error are large. Any excess of metal thus allowed can be trimmed off when the object is formed, whereas a shortage of material would cause the entire piece to be wasted.

105. To develop a right or oblique pyramid

Analysis. Each face of a pyramid is a plane triangle. If the true lengths of the three sides of each triangle are obtained, the triangles may be laid out and joined one to another on a single plane with a common vertex, in the order of their position on the pyramid, to give the required development. If the pyramid is to be closed at the base, this portion of the inclosing surface may be added to the base-line development of any one of the triangular sides.

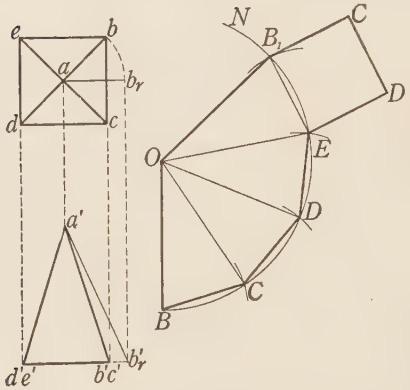


FIG. 142. Development of right pyramid

Construction One. Given the right pyramid A-BCDE, Fig. 142. Begin the development on any lateral edge line, such as AB. The true length of AB, by construction, is $a'b_r'$. All other lateral edges of the pyramid are of this length. Assume the point *O* at any convenient place on the paper as the development of the apex of the pyramid. Draw the line *OB* in any direction and equal to $a'b_r'$ in length. With *O* as a center and a radius *OB*, draw the indefinite arc *BN*. With the point *B* as a center and radius equal to *bc*, the true length of the base of the triangle ABC, draw an arc intersecting the arc *BN* at *C*. Join the point *C* to the points *B* and *O* to obtain the development of the triangle ABC. In a similar manner, the other three triangular faces of the pyramid are laid out in order, to obtain the required development of the lateral faces of the pyramid. The base of the pyramid, BCDE, which shows in true size in the plan view, is added to the developed base of any triangle with due regard to the most economical pattern when used in cutting the shape from specified material.

Construction Two. Given the oblique pyramid A-BCDE, Fig. 143. Begin the development of the pyramid on any lateral edge line of any triangular face, such as AB, for example. The true length of the line AB is $a'b_r'$, by construction; the true length of the line AC is $a'c_r'$, by construction; the true length of BC is shown directly in the plan view as *bc*. Assume the point *O* of the development at any convenient place on the paper. Draw *OB* in any direction and equal to $a'b_r'$ in length. With *O* as a center and a radius equal to $a'c_r'$, draw the indefinite arc *NM*. With *B* as a center and a radius equal to *bc*, draw an arc intersecting *NM* at the point *C*. Connect the point *C* to the points *O* and *B* by straight lines. The triangle *OBC* is the development of the triangular face ABC of the pyramid. Proceed in a similar manner with the other three faces of the pyramid and add the base of the pyramid to the base of any developed triangle, as in Construction One.

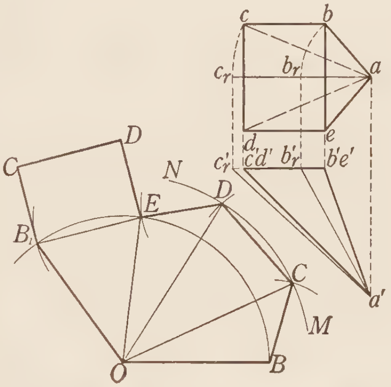


FIG. 143. Development of oblique pyramid

Discussion. The bases of pyramids, prisms, cones, and cylinders are seldom shown in the developments of the surfaces, since the patterns are used mostly in constructing sheet-metal conductors of gases, fluids, and the like.

106. To develop an oblique cone and to trace upon the development the line of intersection of the cone with any other surface

Analysis. If a pyramid of a sufficiently large number of sides is inscribed in the cone, and the pyramid is then developed as in Article 105, the development of the substituted pyramid will be approximately the true shape of the required development of the cone. Close approximation may be attained by increasing the number of sides of the pyramid to the limit of accurate construction possibilities.

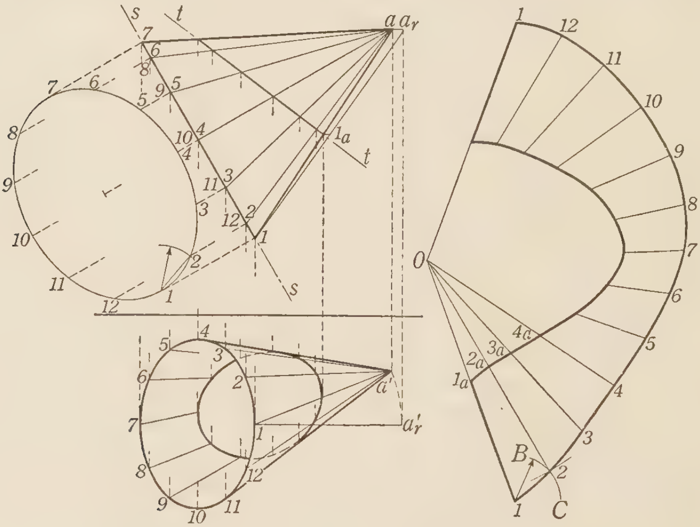


FIG. 144. Development of oblique cone

Construction One. Given the oblique cone A and its line of intersection with a plane already traced upon it, Fig. 144. Divide the base of the cone into twelve or more parts and connect each division point to the adjacent point by a straight line, thus giving the chords 1-2, 2-3, 3-4, etc. One chord only is shown on the drawing. Join each division point on the base to the apex of the cone by elements A1, A2, A3, etc. These elements are the lateral edge lines of the inscribed pyramid. Each chord, together with the two connecting elements, forms a plane triangle, which approximates a corresponding portion of the cone in the development. Pass the plane S through the base of the cone and revolve the base into the horizontal plane about the trace *ss*. This revolved base shows its true shape and therefore the true length of each selected chord, that is, the true length of the base of each triangular face of the inscribed pyramid. Begin the development with any edge line of the pyramid. Assume some point *O* on the paper to represent the vertex of the pyramid on the development. Draw *O1* in any direction and equal in length to a_r1, the true length of element A1 by construction. With the point *1* on this developed element as a center and with a radius equal to the length of the chord *1-2* on the revolved base, describe the indefinite arc *BC*. With the point *O* as a center and with a radius equal to the true length of the element A2, describe an arc cutting the arc *BC* at the point *2* in the development. Connect points *O* and *2* by a straight line. Proceed similarly for successive triangles around the entire pyramid. Join the points *1*, *2*, *3*, etc. on the development by a smooth curve. Lay off from *O*, on each developed element, the true distance of the apex of the cone from the given line of intersection as measured on the true-length projections of the corresponding elements. For element A1 this distance is $O1_a$, equal to a_r1_a on the true-length projection of the element; for element A2 it is $O2_a$; and so on for each of the twelve elements. Connect the points 1_a, 2_a, 3_a, etc. by a smooth curve to complete the development.

Discussion. The true length of each chord might have been obtained from the plan and elevation views in the same way as the true length of the elements, but the method of finding the true shape of the base is preferred not only because it is more accurate, but also because it furnishes a second important item in development work. In rolling a piece of material, which has been cut to the shape of any developed pattern, into the shape of the required cone, it will be found that a pattern or template of the base is necessary to insure that the contour of the cone is correct when the rolling process is completed. This template is made directly from the true-size projection of the base, which, in the foregoing construction, was obtained before the development was begun. The plane-T section may be treated similarly.

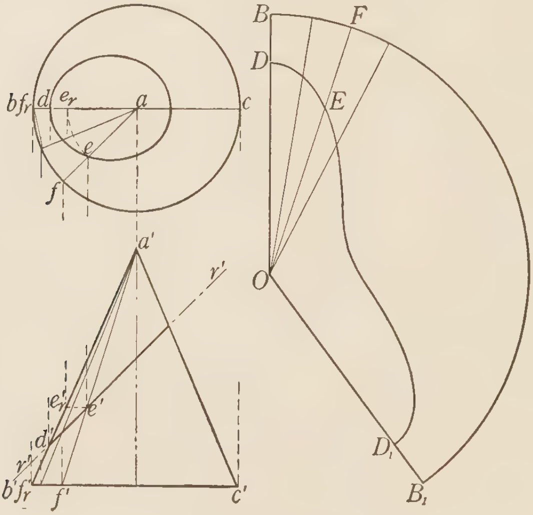


FIG. 145. Development of right circular cone

In the case of a right circular cone, all the elements are equal in length and the chords of the circular projection of the base may also be made equal. This makes it possible to obtain the development of the cone very quickly. It is a sector of a circle with a radius equal to the length of the elements and an arc length equal to the circumference of the base of the cone (see Fig. 145).

Construction Two. In practical work the frustum of a cone is usually shown on the drawing instead of the full cone. The development of the frustum may be accomplished without locating the apex of the cone, by the method of triangulation, so-called. The method consists in substituting a series of plane triangles for the curved surface of the frustum and then developing the triangles in order. The vertex of the first substituted triangle is placed in one base of the frustum, while the base of the triangle is made a chord of the second base of the frustum. One side of the triangle must be an element of the cone. In the second triangle the conditions are reversed, and so on alternately around the entire surface of the cone. This means that both bases must be divided into the same number of proportionate parts, beginning on a projected contour element, in order to obtain an element as one side of each triangle.

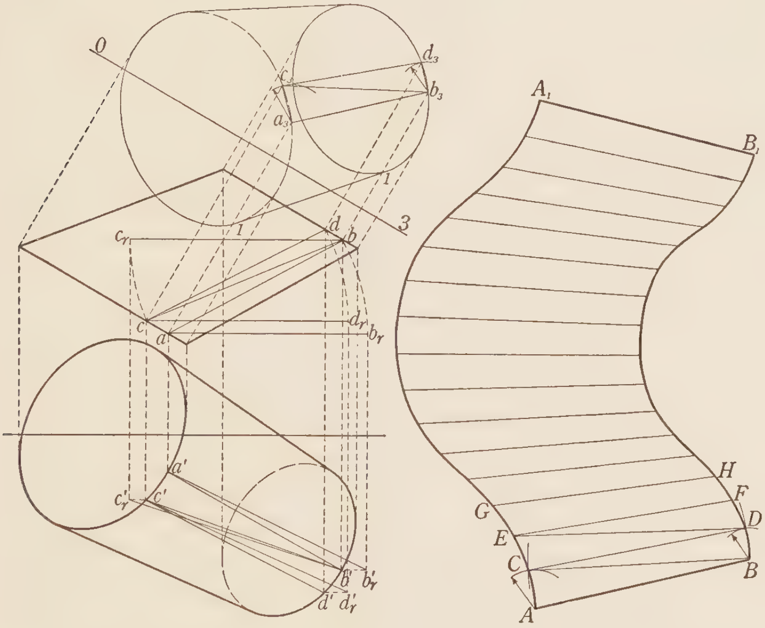


FIG. 146. Development of oblique cone (triangulation method)

Given the frustum of an oblique cone, Fig. 146. Draw the series of alternately reversed triangles, ABC, CBD, DCE, etc., completely around the frustum. The division points on the bases have been obtained from the auxiliary view, starting with contour element 1, as shown. Develop each triangle by the true-size method employed in developing the pyramid of Fig. 143 and the cone of Fig. 144. Only the constructions for the triangles ABC and BCD are shown. A continuous smooth curve is drawn through the points *A*, *C*, *E*, *G*, etc., and also through points *B*, *D*, *F*, *H*, etc., to more closely approximate the curved bases of the frustum when the material used is rolled into shape. Close approximation to the actual development of the frustum is obtained by using a sufficiently large number of triangles. Numerals may be used to advantage to mark the ends of the elements on the projections and the development.

107. To develop an oblique cylinder and trace upon the development the line of intersection of the cylinder with any other surface

Analysis. A right section of any cylinder will roll out, or develop, as a straight line, called the "stretch-out" line. This line will be equal to the true length of the right-section curve. The elements, being perpendicular to the plane of the right section, will develop perpendicular to the "stretch-out" line and in true length. Their points of intersection with this line will be the developed points of intersection of the elements with the right section of the cylinder before it was rolled out flat. A part of any element may fall on one side of the "stretch-out" line, while the remainder may be on the other side, depending upon where the right section is taken.

Construction One. Given the oblique cylinder A and its line of intersection with two planes, Q and R, already traced upon it, Fig. 147. Pass the plane S through the cylinder and perpendicular to its elements to obtain the right section, as shown in the figure. Revolve this section about *ss* into the

H plane to obtain its true size. Divide the true-size projection of the right section into twelve or more parts to insure that the resulting chords closely approximate the arcs they subtend. Draw the H and V projections of the elements of the cylinder through projected points on one of its bases which correspond to the end points of the chords on the true-size projection of the right section. Lay off on the chosen "stretch-out" line, *MN*, the successive chord

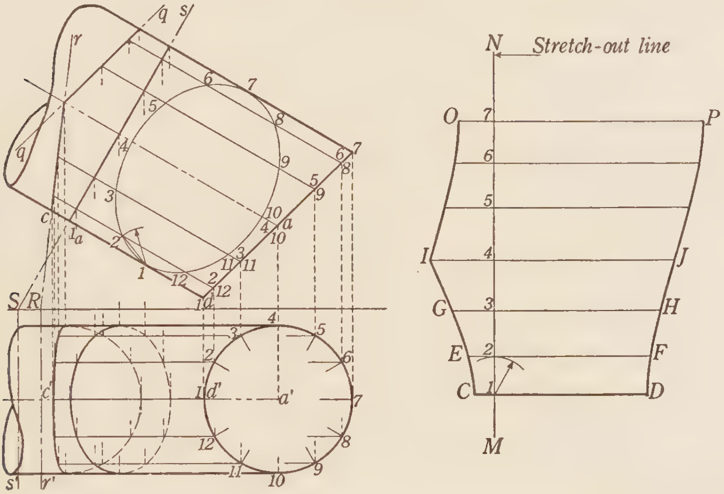


FIG. 147. Development of oblique cylinder

distances *1-2, 2-3, 3-4,* etc. and erect perpendiculars to this base line through each division point. On the perpendicular and to the left of the "stretch-out" line, lay off the length *1-C* equal to the true length of element 1 from the right section to its intersection with plane R, that is, $1_a c$. On the right lay off the length 1-*D* equal to $1_a d$. Proceed similarly for each element of the cylinder. Join the points *C, E, G,* etc. and the points *D, F, H,* etc. by smooth curves. Since the cylinder is symmetrical about a plane passed through elements 1 and 7, only half of the cylinder need be developed. The half pattern may then be used by the shop man in laying out his work, or it may be used by the draftsman in constructing the full pattern.

Discussion. In selecting the element along which the cylinder is to be opened, it is to be kept in mind that the resulting pattern should be as regular in outline as possible in order to prevent waste in cutting the shape from the material to be used in making the cylinder. The method of finding the true size of a right section in Construction One is equivalent in results to projecting the cylinder upon an auxiliary coördinate plane 3 perpendicular to the axis of the cylinder. This can be done by using points on the base without finding the right-section projections at all. A second auxiliary plane is necessary if the cylinder is inclined to all the coördinate planes, as in Construction Two.

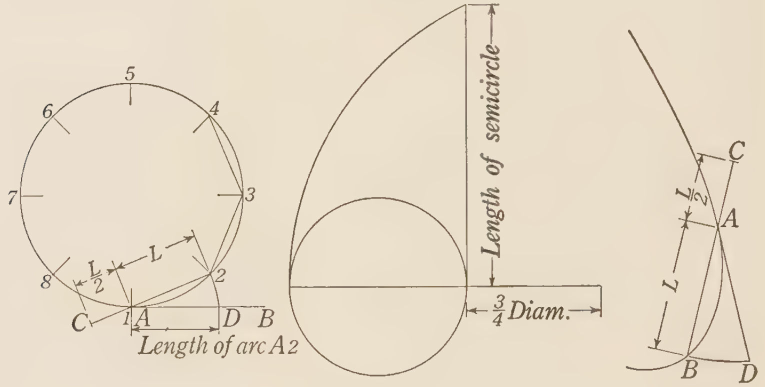


FIG. 148. Rectification of curved lines

The rolling out of a plane curve into its equivalent straight-line length is known as *rectifying* the curve. If greater accuracy is desired in obtaining the "stretch-out" line than is afforded by the substitution of chord distances for their subtended arcs, each arc may be rectified by the graphical method shown in Fig. 148, and these rectified lengths used in laying out the pattern. In the first drawing of the figure the circle to be rectified is divided into a number of equal arcs, *1-2, 2-3, 3-4,* etc., by means of the corresponding equal chords, subtending equal angles. At the point *A* on the first arc a tangent, *AB,* is constructed. The chord *1-2* is then extended outside the circle a distance equal to one half its length to the point *C*. With the point *C* as a center and a radius equal to *C-2,* the arc *2-D* is described. The line *AD* is very closely equal in length to the arc *1-2*. This method of rectification gives accurate results on circular arcs with subtended angles close to 60°. For larger angles the chord extension must be increased, until at 180° it becomes approximately equal to three fourths of the length of the chord (diameter), as shown in the second drawing of the figure. For angles smaller than 60° the chord extension should be decreased. At 30° the extension is approximately three eighths of the chord length. Using a chord extension of one half, subtended angles between 30° and 60° may safely be chosen.

The advantage of the rectification construction given in the preceding paragraph over the straight-chord method of determining the length of a given circle lies only in the fact that a longer arc distance may be chosen as the unit length with which to obtain the total length of arc, thus eliminating the accumulated error of many short measurements. For ordinary purposes the chord method is sufficiently accurate. The rectification method may be applied to arcs that are not parts of circles, with a resulting degree of accuracy depending upon the chord distances selected.

Construction Two. Given the oblique cylinder B with both bases plane, but not right sections of the cylinder, Fig. 149. Project the cylinder upon plane 3, which is set up parallel to the elements of the cylinder. This is done by first projecting either numbered base upon the plane, then drawing the elements through the projected base points parallel to the projected axis of the cylinder, and finally projecting the second base points upon the corresponding elements.

Project the cylinder upon plane 4, which is set up perpendicular to the elements of the cylinder. This is done by using one of the numbered bases. The latter auxiliary projection coincides with the plane-4 projection of any and all right sections of the cylinder. Pass the right-section plane S through the cylinder in the plane-3 view only. This is the plane in which the "stretch-out" line is to be taken and from which measurements along the elements may be made to each base. Develop the cylinder from this point on in the same way as was the cylinder of Fig. 147. The several lengths, *1-2*, *2-3*, etc., on the chosen "stretch-out" line are taken directly from the plane-4 view as chords of subtended

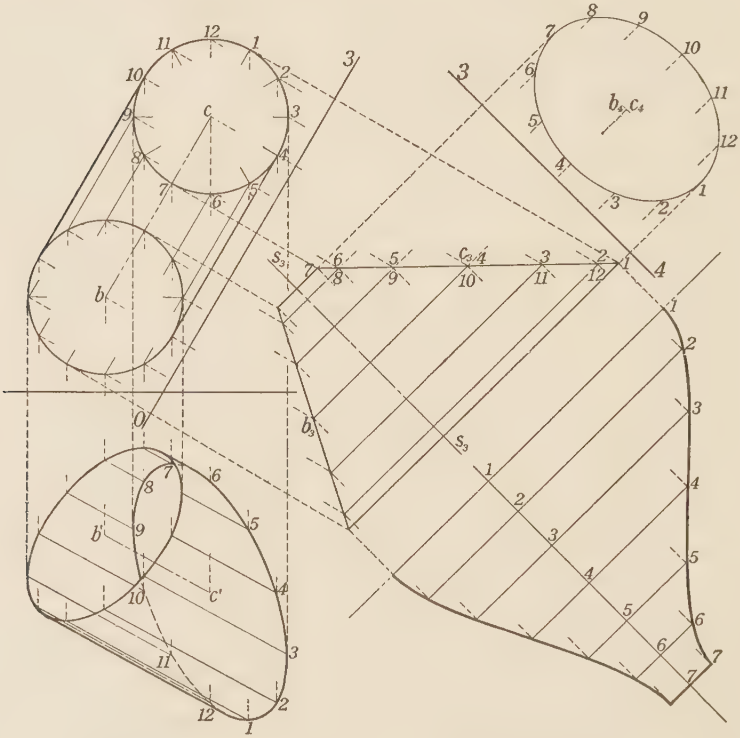


FIG. 149. Development of oblique cylinder

arcs. In very accurate work each chord should be rectified by the method of Fig. 148. Since the cylinder is symmetrical about a plane passed through elements 1 and 7, only a one-half pattern is required.

Construction Three. The triangulation method employed in developing the frustum of the oblique cone of Fig. 146 may be used equally well with the oblique cylinder. The procedure is illustrated in Fig. 150. Reference should be made to Construction Two of Article 106 for an explanation of the details of construction. This method is not recommended for general use because of the impossibility of obtaining as accurate patterns in this way as by other methods.

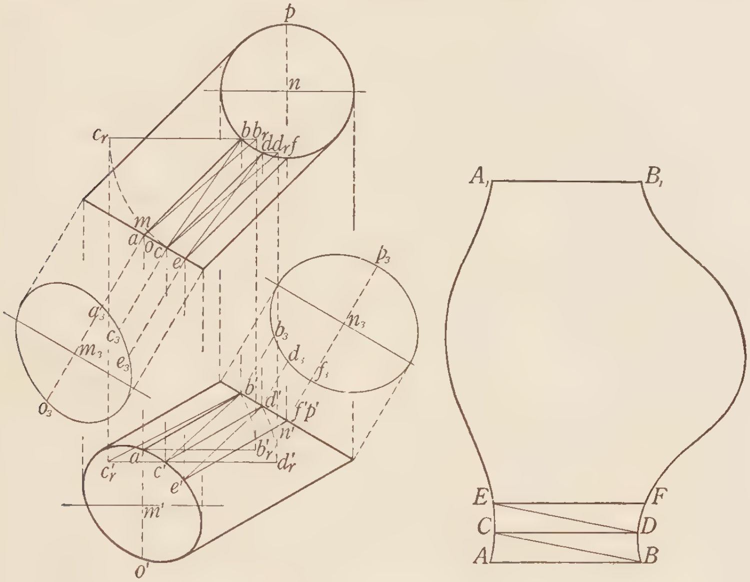


FIG. 150. Development of oblique cylinder (triangulation method)

108. Approximately developable surfaces

Many transition pieces used in engineering and industrial design cannot, in the strict geometrical sense, be developed. They may, however, so closely approximate one or another

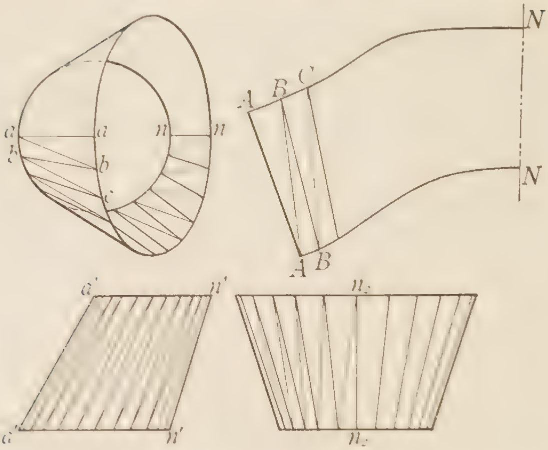


FIG. 151. Approximately developable surface

of the developable surfaces that a pattern can be made by the triangulation method employed with the cone and cylinder in Figs. 146 and 150, or by some other approximate method of development that will insure the form or shape being pressed or rolled from sheet metal with the minimum stretching and crimping.

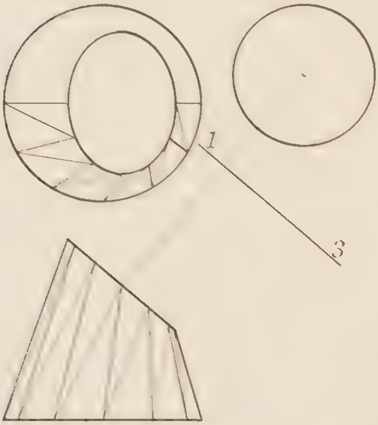


FIG. 152. Approximately developable surface

The transition piece shown in Fig. 151 joins an elliptical to a circular opening, with the connections in parallel planes. The simplest way to generate this surface is by moving a straight line along the two bases, beginning at the element of symmetry AA, in such a way that the rates of motion of the points of contact along the two bases are proportional, at any instant, to the respective radii of curvature. This transition piece is a warped surface. Its approximate development is obtained by drawing several elements, AA, BB, etc., and the corresponding chords on the bases to form a series of warped quadrilaterals which closely approximate the generated surface. The diagonals AB, BC, etc., are then drawn between the elements, to form a series of plane triangles. These triangular segments are laid out in the proper sequence to form the complete development of the surface. The constructions are shown for the first two quadrilaterals only.

The reducing section of Fig. 152 is treated in a similar manner. Its two bases are circles of different radii, and they do not lie in parallel planes. The surface is a warped surface not greatly different from an oblique cone (see Fig. 110). Several elements are shown, together with the diagonals, which reduce the entire piece to triangular segments which may be developed as were those of Fig. 151.

CHAPTER VI

THE RIGHT CIRCULAR CONE AS A CONSTRUCTION AID

109. The Θ, Φ, and Π cones

When a right circular cone, as defined in Article 87, is turned so that its axis is perpendicular to the H plane, it is called a Θ cone and its base angle is marked Θ. When its axis is turned perpendicular to the V or P plane, it is called a Φ or Π cone, and its base angle is marked Φ or Π accordingly. If a point O, chosen anywhere in space, is made the apex of one of these cones, say a Θ cone, for example, then the successive elements of the cone, or what amounts to the same thing, the cone itself, becomes the locus of a line in space which passes through the point O and makes an angle with the horizontal plane equal to the base angle, Θ, of the cone. In like manner, a Φ cone, with its apex at any chosen point in space, is the locus of a line passing through the point and making an angle with the vertical plane equal to the base angle, Φ, of the cone. Similarly, a Π cone is the locus of a line through a point making a specified angle, Π, with the profile plane. Hence the Θ, Φ, and Π cones may be used to draw lines in space making given angles with the coördinate planes and, at the same time, satisfying other possible conditions which may be set up concerning the lines.

Any plane tangent to a Θ cone will make an angle Θ with the H plane equal to the base angle of the cone. If tangent to a Φ or Π cone, the plane will make the angle Φ or Π with the V or P plane. Therefore the Θ, Φ, and Π cones can be used as construction cones for passing planes through a point and making given angles with the coördinate planes. Each plane so passed will contain a line tangent to the base circle of the construction cone in addition to an element of the cone.

110. To find a second projection of a line when one point on the line, one projection of the line, and any one of the angles Θ, Φ, or Π are given

Analysis. With the given point as an apex, draw a Θ, Φ, or Π cone, according to the angle specified in the problem, with its base angle equal to the given angle and its altitude of any length. The required line is an element of the cone and may be determined directly.

Construction. Given the point O, the vertical projection $o'a'$, and the angle $\Theta = 45°$, of the required line OA, Fig. 153.

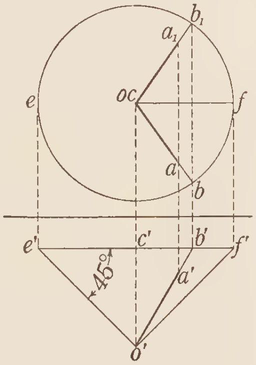


FIG. 153. A Θ construction cone

With O as the apex, construct the Θ cone with its axis, OC, perpendicular to the horizontal plane and its base angle equal to 45°. The vertical projection of the cone is the isosceles triangle $o'e'f'$. Its horizontal projection is the circle whose center is o with diameter equal to $e'f'$. The line OA, being an element of the cone, intersects the base at some point B, the vertical projection of which is found by extending $o'a'$ until it intersects the vertical projection of the base at b'. The H projection of the point B is on the circle directly above b' at b or b_1, there being two solutions to the problem under the given conditions. The H projection of OB being determined, the point a or a_1 is located upon it in the usual manner.

Discussion. It should be observed from the figure that it would have been impossible to find a solution for the problem had $o'a'$ fallen outside the isosceles triangle $o'e'f'$. Also, had the H projection of OA been given in the problem data instead of the V projection, the horizontal projection, b, of the point B would have been determined first, as the intersection of the horizontal projection of the base of the cone (a circle) and the horizontal projection of the line. Further, it should be remembered, in any case, that the construction cone must correspond in name to that of the angle given in the problem specifications. Adherence to this rule will materially aid in determining the projections of the cone to be drawn. The student should obtain the solutions for all possible cases in which the angles Φ and Π may be given and which require, therefore, that Φ and Π cones be drawn in the constructions.

111. To draw a line through a given point, making two specified angles, Θ, Φ, or Π with any two corresponding coördinate planes

Analysis. If the given point is made the apex of two right circular construction cones (Θ, Φ, or Π cones according to the angles specified in the problem data) with their base angles corresponding to the given angles, either of the two common elements of the cones will be the required line.

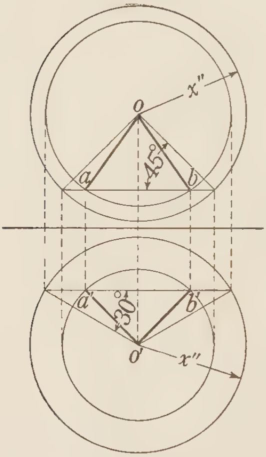


FIG. 154. The Θ and Φ construction cones

Construction. Given the point O and the angles $\Theta = 30°$ and $\Phi = 45°$, Fig. 154. With the point O as the apex, construct a Θ cone with its base angle equal to 30° and its elements of any chosen length. With the same apex O, construct a Φ cone with its base angle equal to 45° and its elements equal in length to those of the Θ cone. These constructions put the apex of each cone at the center of a sphere and the base circle of each cone on the surface of the sphere, the radius of the sphere being equal to the chosen length of elements of the two cones, as shown in the figure. The sphere thus becomes a locus surface for the bases of the two cones, which makes it possible to locate two points that are common to the two cones, in addition to the point O at the center of the sphere. These points are A and B, at the intersection of the bases of the cones. The lines OA and OB, being elements common to the two cones, that is, being the lines of intersection of the cones, make the required angles with the H and V planes.

Similar constructions are necessary when Θ and Π or Φ and Π are given.

Discussion. If the sum of the two specified angles is 90°, the two construction cones will be tangent and there will be only one solution; if the sum of the two angles is more than 90°, there is no solution.

112. To draw a line in a given plane and making a given angle Θ, Φ, or Π with the corresponding coördinate plane

Analysis. If any point O is chosen in the given plane and made the apex of a Θ, Φ, or Π cone, according to the angle specified in a particular problem, either of the lines cut from the cone by the plane — one line when the plane is tangent to the cone — will be the required line.

Construction One. Given the plane S and the angle $\Theta = 45°$ of a line required to be drawn in the plane S, in the first drawing of Fig. 155. Assume any point O in the plane S and construct a Θ cone with its apex at O, its base angle equal to 45°, and its base resting in the H plane. The H plane contains all the points that are on the base of the cone. The H trace of the plane S contains all the points in the H plane that are also in the plane S. Therefore, the points A and B, coinciding with their own H projections, at the intersection of *Ss* with the base circle, are points common to the plane and cone. The lines OA and OB lie wholly in the plane S, and make an angle of 45° with the H plane, since both are elements of the Θ cone. Either may be taken as the required line.

Construction Two. Given the plane T and the angle $\Theta = 45°$ of a line required to be drawn in the plane T, in the second drawing of Fig. 155. Assume the point O in the plane T and construct a Θ cone with its apex at O, its base angle equal to 45°, and its elements of any length. Pass the plane R through the base of the cone. The plane T intersects the plane R in the line V_1D, which is the locus of all points that are common to the two planes. The line V_1D intersects the base circle of the cone in the points E and F, two points that are common to the plane T and the cone. The lines OE and OF lie wholly in the plane T and make an angle of 45° with the H plane, since both are elements of the Θ cone. Either may be taken as the required line.

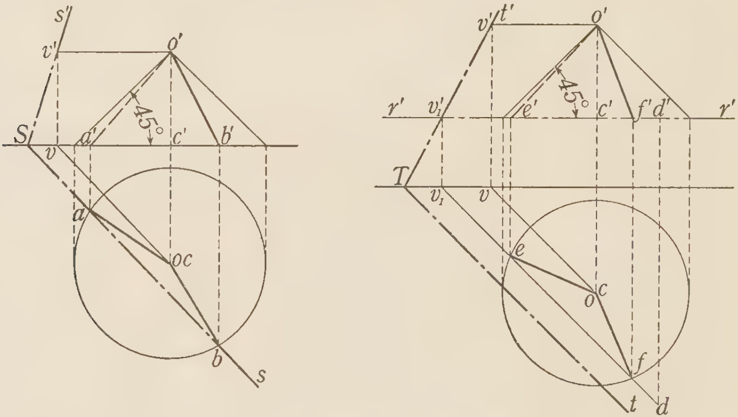


FIG. 155. Specified line in a plane (cone method)

Discussion. If the angle Φ or Π is given, then a Φ or Π cone must be drawn with its apex in the given plane, and the construction procedure must be carried out in a manner similar to that of Construction One or Two.

113. To find the angles Θ, Φ, and Π of a given plane

Analysis. If any point in the given plane is successively made the apex of a Θ, Φ, and Π cone, and if the base circle of each cone is drawn in a coördinate plane and made of such a size that it is tangent to the corresponding trace of the given plane, then the constructed cones will be tangent to the given plane and their respective base angles will be the measures of the required angles.

Construction One. Given the plane S in the first drawing of Fig. 156. Assume the point O in the given plane as the apex of a Θ cone with its base resting in the H plane. The H projection of the base of the cone is the circle *abd* with its center at *o* and tangent to *Ss* at *d*. The V projection of the base of the cone is *a′b′* in the horizontal ground line. The vertical projection of the constructed cone is *o′a′b′*. The cone is tangent to the plane along the element OD. The base angle Θ of the cone is the angle the plane S makes with the H plane; it is, therefore, one of the required angles. Similar constructions, using Φ and Π cones, will determine the other required angles, Φ and Π respectively.

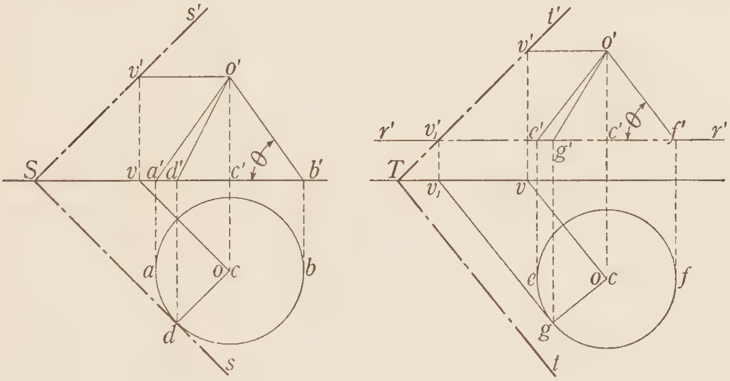


FIG. 156. Angle of a plane (cone method)

Construction Two. Given the plane T in the second drawing of Fig. 156. Assume the point O in the given plane as the apex of a Θ cone whose elements are of any length and whose base is drawn in a plane R parallel to the H plane. Find the line of intersection, V_1G, of the planes R and T. Construct the base circle of the cone tangent to the line

V_1G. In horizontal projection, the base circle is tangent to v_1g and has its center at o; in vertical projection the circle appears as $e'f'$ in the trace $r'r'$. The vertical projection of the constructed cone is $o'e'f'$. The cone is tangent to the plane T along the element OG. The base angle Θ is the angle the plane T makes with the horizontal plane; it is, therefore, one of the required angles. Similar constructions, using Φ and Π cones, will determine the other required angles, Φ and Π respectively.

114. To find a second trace of a required plane when one trace and one of the angles Θ, Φ, or Π of the plane are given

Analysis. If a Θ, Φ, or Π cone, corresponding to the angle specified in a given problem, is constructed with its base resting in that coördinate plane to which its axis must be perpendicular by definition, and with its apex in that coördinate plane on which a trace of the required plane is either given or to be drawn, then a plane, having one of its traces passing through the apex of the cone and another trace tangent to the circular projection of the base circle, may be passed which will be parallel to the required plane and the direction of the second required trace thus determined. If the apex of the constructed cone is placed in the given trace, or the circular projection of the base is made tangent to the given trace, depending upon the problem specifications, the work of construction may be shortened materially and the solution made directly without obtaining a direction plane to which the required plane is parallel.

Construction One. Given the vertical trace Ss' and Θ equal to any specified angle, in the first drawing of Fig. 157. Assume o' as the vertical projection of any point O in the trace Ss' with the H projection of O in the ground line, and construct a Θ cone with O as its apex, its base angle equal to the specified value of Θ, and its base resting in the H plane.

The isosceles triangle $o'a'b'$ is the vertical projection of the cone. The circle whose center is o is the H projection of the base of the cone and coincides with the base itself. Draw the required trace, Ss, tangent to the H projection of the base circle at c, in either of the two possible directions. The plane S is tangent to the cone along the element OC.

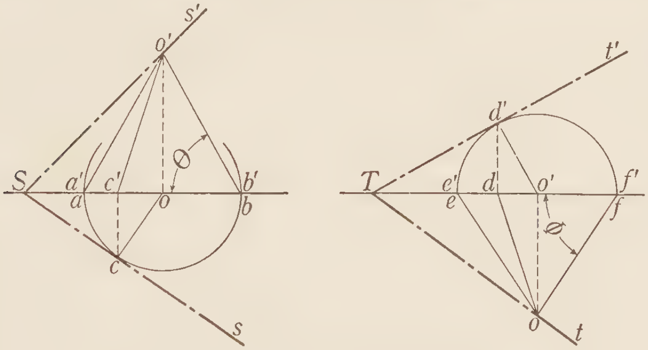


FIG. 157. Traces of specified planes (cone method)

It is evident from the drawing that Θ cannot be smaller than the angle which the vertical trace makes with the horizontal ground line. If Θ is equal to this angle, there is only one solution.

Construction Two. Given the vertical trace Tt' and Φ equal to any specified angle, in the second drawing of Fig. 157. Assume o' in the horizontal ground line as the vertical projection of any point O in the required H trace of the plane T and construct a Φ cone with O as the apex, its base angle equal to the specified value of Φ, and its base resting in the V plane. The circle whose center is o' and which is described tangent to the trace Tt', is the V projection of the base of the cone and coincides with the base itself. The isosceles triangle oef, with its base angle equal to the specified value of Φ, is the H projection of the cone. The vertex o of this triangle is the horizontal projection of the apex of the Φ cone and coincides with the apex itself. Draw the required trace *Tt* through the points *T* and *o*. The plane T is tangent to the cone along the element OD.

It is evident from the drawing that the base angle of the construction cone cannot be greater than 90°, hence when Φ is more than 90° the supplementary angle must be used, and further that the cone can be drawn with its apex behind the vertical plane, thus permitting two solutions.

The solutions of this problem when other combinations of traces and angles are given are similar to those of Constructions One and Two.

Discussion. Right circular cones employed in the manner of this article are called construction or locus cones for determining oblique planes. Their special location in the above constructions makes it possible to secure solutions with the minimum effort. The apex, as well as the base, could have been selected out of the coördinate planes and the constructions carried out in the manner of those of the second drawings of Figs. 155 and 156. The right circular cone may be used as a construction cone under very general conditions; for example, in the passing of a plane through a point and making a specified angle with an oblique plane.

115. To pass a plane through a given point, making a given angle Θ, Φ, or Π with the H, V, or P plane respectively

Analysis. Any number of planes may be passed satisfying the given conditions. If the given point is made the apex of a Θ, Φ, or Π cone, depending upon the given angle, any plane tangent to the cone will be the required plane.

Construction One. Given the point O and Θ equal to any specified angle, in the first drawing of Fig. 158. With the point O as an apex, construct a Θ cone with its base angle equal to the given angle Θ and its base resting in the H plane. The vertical projection of the cone is the isosceles triangle *o′a′b′*; the H projection is the circle whose center is *o* and whose diameter is equal to *a′b′*. Draw the H trace, *Ss*, of the required plane tangent to the circle at any point *c*. The plane S must be tangent to the cone along the element OC; therefore the V trace of the plane S must pass through the V piercing point of the line OC. Draw *Ss′* through the points *S* and *v′*. Plane S is one of many planes satisfying the given conditions.

Construction Two. Given the point O and Θ equal to any specified angle, in the second drawing of Fig. 158. With O as an apex, construct a Θ cone with its base angle equal to the given angle Θ and its base resting in the H plane. Draw the H trace, *Tt*, of the required plane tangent to the circular projection of the base at any point. Since the required plane must contain the apex of the cone, it must contain an H- or a V-parallel drawn through the point O, in this case the line OV. Draw *Tt′* through the points *T* and *v′*. Plane T is one of many planes satisfying the given conditions.

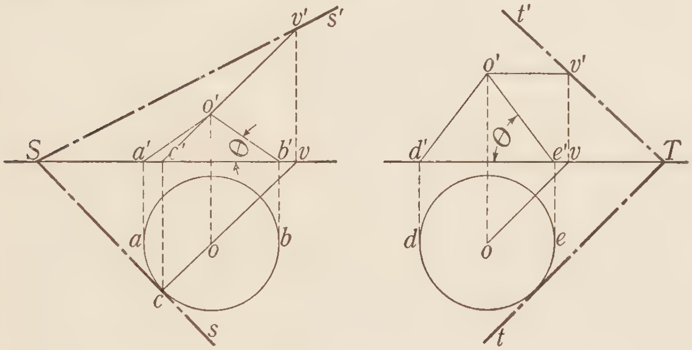


FIG. 158. Specified planes determined (cone method)

116. To pass a plane through a given point, making two specified angles, Θ, Φ, or Π, with any two corresponding coördinate planes

Analysis. If a plane is passed perpendicular to a line that makes a given angle with a coördinate plane, the plane will make the complementary angle with the coördinate plane.

Hence, if a line is drawn through any chosen point in space making angles with the coördinate planes that are complements of the given angles, then a plane passed through the given point perpendicular to this line will be the required plane.

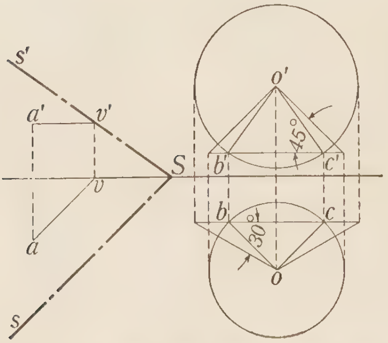


FIG. 159. A specified plane determined (cone method)

Construction. Given the point A, the angle Θ = 45°, and the angle Φ = 60°, Fig. 159. Assume a point O in any convenient position in space and draw the lines OB and OC so that Θ of both equals 90° – 45° = 45° and Φ equals 90° – 60° = 30°, by the Construction of Article 111. Plane S, passed through the point A perpendicular to either OB or OC by the Construction of Article 50, is the required plane.

117. To pass a plane through a given line and making a given angle Θ, Φ, or Π with the H, V, or P plane respectively

Analysis. If any point in the line is made the apex of a Θ, Φ, or Π cone, depending upon the angle given, the plane passed through the given line and tangent to the construction cone is the required plane. The given line may not make an angle with the coördinate plane to which the axis of the cone must be drawn perpendicular greater than the given angle.

Construction. Given the line AB and the angle Θ = 60°, Fig. 160. Select any point, such as B, on the given line and construct a Θ cone with its apex at B, its base angle equal to 60°, and its base resting in the H plane. Draw the H trace, *Ss*, of the required plane through *h* tangent to the circular projection of the base of the cone at either of the two possible points. Draw Ss' through the points S and v'. Plane S is the required plane. It is tangent to the cone along the element BC.

If the angle Φ or Π is given, a Φ or Π cone should be drawn and the corresponding piercing point of the line used in drawing the first trace of the plane.

Discussion. If the circular projection of the base of the cone passes through h, there is only one solution. If h falls within the base circle, there is no solution.

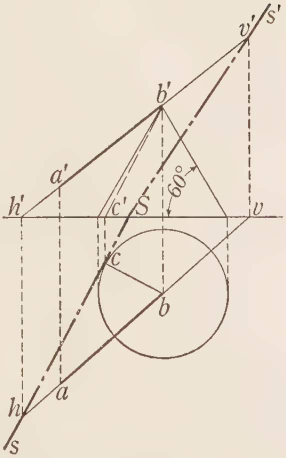


FIG. 160. A specified plane determined (cone method)

118. Planes tangent to surfaces

In Articles 114, 115, and 117 (Figs. 157, 158, and 160), oblique planes were passed tangent to right circular cones whose axes were perpendicular to one of the coördinate planes. Inspection discloses the fact that, in each instance, the tangent plane contains two intersecting lines that were, or could have been, used in the construction, one being an element of the cone, the other being tangent to the base of the cone and lying in a coördinate plane. This latter line was not shown in the constructions, since it coincided with the trace of the required plane. In passing a plane tangent to a ruled surface, in any position in space, it is generally possible to fix the location and tilt of the plane by means of two similar straight lines, one drawn on the surface as an element, the other drawn tangent to the surface at some point on the first line.

A plane is tangent to a surface of revolution at a point on the surface when it contains two lines that are drawn tangent to two curves on the surface passing through the point.

In passing a plane tangent to any surface of revolution through a given point or line, it is often advantageous to employ the right circular cone as a construction device, because of the ease with which one or both of the two necessary tangent construction lines can be drawn with its aid.

In the following articles the application of these two general schemes of attack will be exemplified.

119. To pass a plane tangent to a given cone or cylinder through some point on the surface of either

Analysis. A plane tangent to a cone or cylinder at a given point on the surface contains the element drawn through the given point. It also contains a line drawn tangent to the base curve at the point of intersection of this curve and the located element.

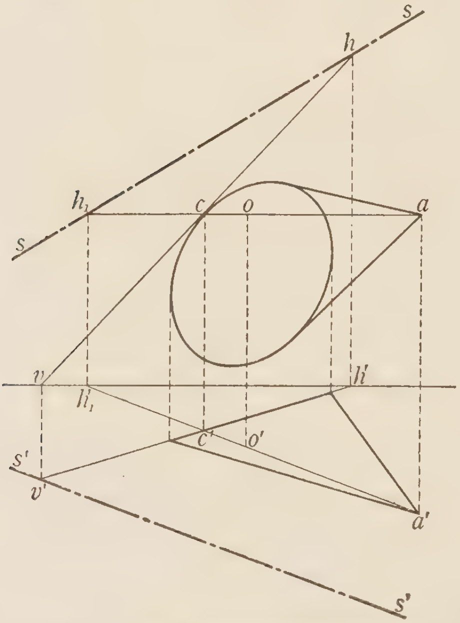


FIG. 161. Plane tangent to cone through point on cone

Construction. Given the cone A and the point O on it, Fig. 161. Draw the element AC of the cone through the point O. Draw the line VH tangent to the base at the point C (see Article 82). Pass the required plane S through the lines AC and VH.

The construction for a cylinder is similar to that for the cone.

Discussion. The plane of the base of the cone of Fig. 161 is a V projecting plane. When the plane of the base is oblique to all of the coördinate planes it is good practice to determine this plane and to revolve it into a coördinate plane, taking the base curve with it, in order to draw the tangent line, and then to return the constructed tangent to the usual projections. This procedure may be shortened somewhat without serious inaccuracy in the final results by omitting the revolution step altogether and drawing the projections of the tangent line respectively tangent to the projections of the base curve, taking care, nevertheless, to see that the line lies in the plane of the base, as evidenced by its piercing points being in the corresponding traces of this plane.

120. To pass a plane tangent to a given cone or cylinder through a given point outside the surface

Analysis. A plane tangent to a cone or cylinder will contain the apex of the surface and hence the line joining the given point to the apex. The required plane will also contain the line drawn tangent to the base curve through the point in which the first line pierces the plane of the base. The required plane will also contain an element of the surface drawn through the point of tangency of the second line with the base.

Construction. Given the oblique elliptical cone A and the point O in the first drawing of Fig. 162. Connect the apex A of the cone to the point O. The line AO pierces the plane of the base at the point P. This point is obtained without construction, since the plane of the base is a V projecting plane (see Article 52). Draw the line PB tangent to the base curve at the point B. This is done by drawing *pb* tangent to the circular projection of the base at either of the two possible points, and making *p′b′* coincide with the V trace of the projecting plane of the base. Plane S, passed through the intersecting lines AP and PB, is the required plane.

In the second drawing of the figure, the line drawn through the given point O is parallel to the elements of the cylinder, since the apex in this case is at infinity on the axis of the cylinder. Otherwise, the constructions are the same in the two figures.

Discussion. When the plane of the base of the given cone or cylinder is oblique to all the coördinate planes, it is necessary to determine this plane and to find the point P by the general method of Article 52. It should be noted in both of the drawings of the figure that an element drawn through the point of tangency on the base may be substituted in the final step of the constructions for either of the two lines that were used in determining the required plane.

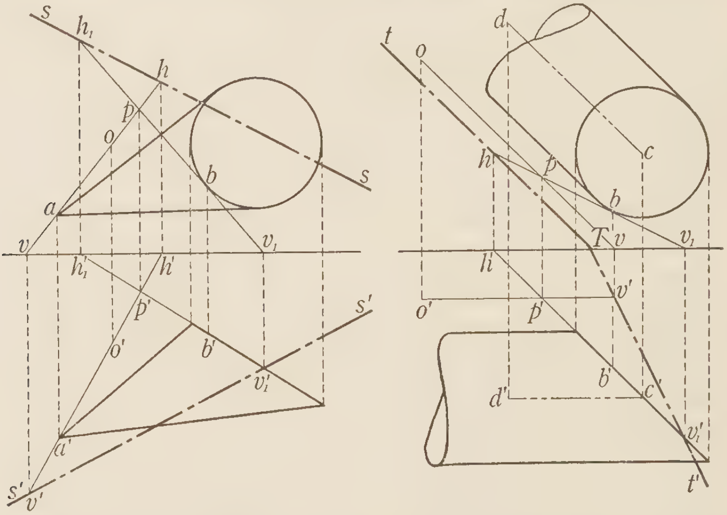


FIG. 162. Plane tangent to cone or cylinder through external point

121. To pass a plane tangent to a given cone and parallel to a given line

Analysis. A plane, to be parallel to a given line, must contain at least one line that is parallel to the given line; to be tangent to a cone it must contain the apex of the cone.

Hence the required plane will contain a line drawn through the apex of the cone parallel to the given line. The plane will also contain a line drawn tangent to the base curve through the point in which the first line pierces the plane of the base.

Construction. Given the oblique elliptical cone A and the line MN, Fig. 163. Through the apex A of the cone draw the line AH parallel to MN. It pierces the plane of the base at the point P. Through the point P draw the line PB tangent to the base curve at the point B. Plane S, containing the lines AP and PB, is the required plane.

Discussion. The element AB may be substituted for either AP or PB in the final step of determining the traces of the tangent plane. If the plane of the base of the cone is inclined to all the coördinate planes, the point P, in which the line drawn through the apex pierces the plane of the base, must be obtained by the general method of Article 52.

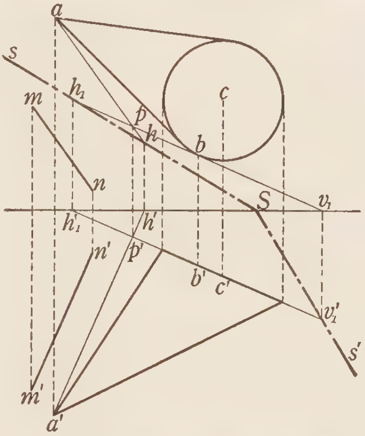


FIG. 163. Plane tangent to cone and parallel to line

122. To pass a plane tangent to a given cylinder and parallel to a given line

Analysis. Since a plane tangent to a cylinder contains one element of the cylinder and is parallel to all the others, the inclination of the required plane can be determined by passing a plane through the given line and a second line drawn parallel to the elements of the cylinder and through some point on the given line. This directing plane and the required parallel tangent plane are intersected by the plane of a base of the cylinder in two parallel lines, one of which is tangent to the base curve. The plane containing this tangent to the base curve and passed parallel to the direction plane is the required plane.

Construction. Given the oblique elliptical cylinder A and the line MN, Fig. 164. From any point O on the line MN draw the line OV parallel to the elements of the cylinder, and pass the plane R through the two intersecting lines MN and OV. Pass the plane S through the base of the cylinder. Its intersection with the plane R is the line V_2H_2. Draw the line H_3C parallel to the line V_2H_2 and tangent to the base curve at the point B. Plane T, passed parallel to plane R and containing the line H_3C, is the required plane.

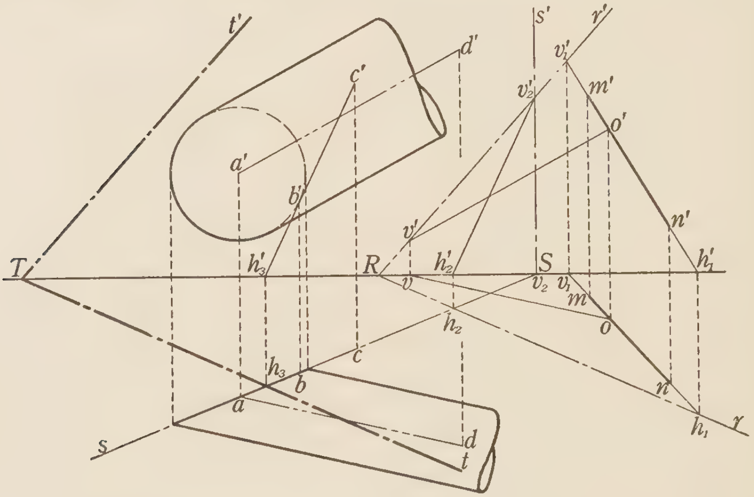


FIG. 164. Plane tangent to cylinder and parallel to line

Plane T can be determined, also, by using the line H_3C and the element intersecting it at the point B. Two solutions are possible, the second being obtained by drawing the tangent H_3C on the opposite side of the base. The directing plane R could have been passed through the axis AD of the cylinder and parallel to the line MN.

123. To pass a plane tangent to a given cone and making a given angle with the H, V, or P plane

Analysis. The required plane must contain the apex of the given cone. If the apex of the given cone is also made the apex of a Θ, Φ, or Π cone, depending upon the angle given, then the plane passed tangent to the two cones will be the required plane.

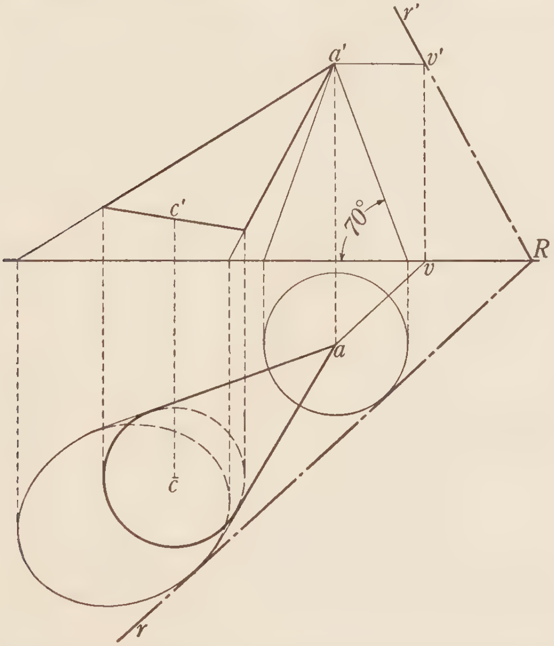


FIG. 165. Plane tangent to cone and with stated inclination

Construction. Given the oblique elliptical cone A and the angle the required tangent plane is to make with the H plane equal to 70°, Fig. 165. Extend the cone A until it intersects the H plane, and determine the new base by finding the piercing points of a number of elements in the plane (see Fig. 131). With the point A as an apex, draw a Θ cone with its base resting in the H plane and its base angle equal to 70°. Any plane tangent to these two cones must have its

H trace drawn tangent to the H projection of the two bases now in the H plane. Draw *Rr* tangent to the elliptical and circular base projections as shown. Since the plane R must contain the point A, it will contain a certain group or "disk" of lines drawn through this point parallel to or intersecting the H trace, *Rr*, of the required plane. Draw *av*, the H projection of the H-parallel AV, parallel to *Rr*. Draw *a′v′* parallel to the horizontal ground line. The line AV pierces the vertical plane at the point V, which is in the vertical trace of the plane R. Draw *Rr′* through the points *v′* and *R*. Plane R is the required plane.

There are four possible locations for the H trace of the plane R in the above construction; hence there are four possible solutions. Fewer solutions are possible when the bases intersect or are tangent to each other.

Similar constructions will determine the required plane when the angles Φ or Π are given, but a Φ or Π cone must be used and the cone must be extended until a new base is found in the V or P plane.

124. To pass a plane tangent to a given cylinder and making a given angle with the H, V, or P plane

Analysis. The required plane will be tangent to the given cylinder along an element of the cylinder and it will be parallel to a directing plane that has been set up making the specified angle with the proper coördinate plane and containing a line parallel to the elements of the given cylinder.

Construction. Given the oblique elliptical cylinder A and the angle the required tangent plane is to make with the H plane equal to 60°, Fig. 166. At any convenient place draw the line BC parallel to the elements of the cylinder. Construct a Θ cone with its apex at some chosen point, such as C, on the line BC, its base angle equal to 60°, and its base resting in the H plane. The plane S, containing the line BC and tangent to the Θ cone (see Article 117) is the required directing plane. Draw the H trace, *rr*, of the required plane parallel to the trace *Ss* of the directing plane and tangent to the base curve of the cylinder at *d*. The required plane must be tangent to the cylinder along the element DE. Draw $r'r'$ parallel to Ss' and through v_1', the V piercing point of the line DE. Plane R is the required tangent plane.

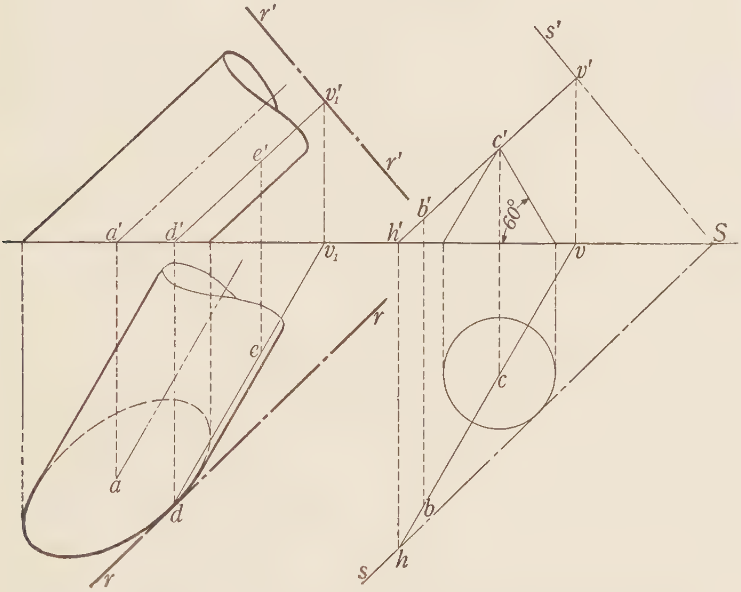


FIG. 166. Plane tangent to cylinder and with stated inclination

The H trace, *Ss*, of the directing plane could have been drawn in either of two possible positions, as could have the H trace, *rr*, of the required plane, thus making four possible solutions, as may be seen from the foregoing constructions. However, if the line BC should coincide with an element of the construction cone, there are only two solutions. If the line BC should fall within the construction cone, there is no solution. If the base of the cylinder is not given in a coördinate plane, the cylinder must be extended and a new base found in the manner shown in Fig. 131 for the cone.

Similar constructions will determine the required plane when the angle Φ or Π is given, but the cylinder must be extended until its base is in the V or P plane and a Φ or Π cone must be used.

125. To pass a plane tangent to a given surface of revolution at a given point on the surface

Analysis. A plane tangent to a surface of revolution at a given point on the surface contains all the lines that can be drawn through the point tangent to the surface. Two of these lines may readily be obtained; namely, one tangent to the curve cut from the surface by a plane passed through the point perpendicular to the axis of revolution, the other tangent to the curve cut from the surface by a plane passed through the point and containing the axis of revolution.

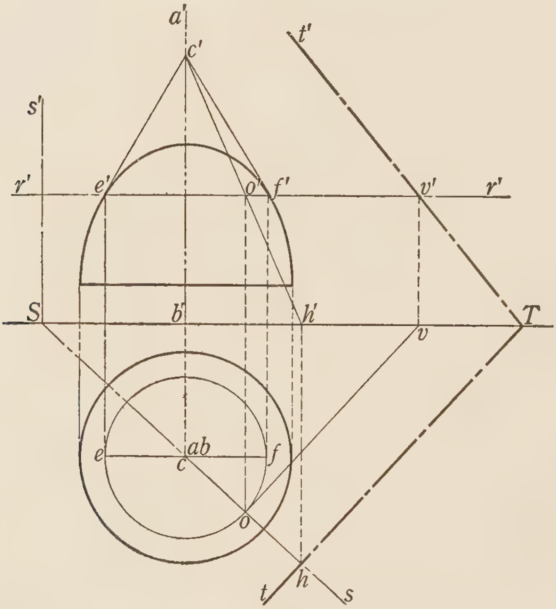


FIG. 167. Plane tangent to ellipsoid of revolution through point on surface

Construction One. Given the half prolate ellipsoid of revolution and the point O on its upper front portion, Fig. 167. Pass the plane R through the point O perpendicular to AB, the axis of revolution of the ellipsoid. It cuts the circle OEF from the ellipsoid. Draw the line OV tangent to this circle at the point O; *ov* is then tangent to the horizontal projection of the circle at *o*, and *o′v′* lies in the V trace, *r′r′*, of the cutting plane. Pass the plane S through the axis AB and the point O to cut a meridian ellipse from the ellipsoid. A line tangent to this meridian ellipse at the point O is an element of a right circular construction cone that is tangent to the ellipsoid on the circle OEF and has its apex in the axis of the ellipsoid. The outside elements, EC and FC, of the tangent cone are located by first drawing their vertical projections, *e′c′* and *f′c′*, tangent to the principal meridian ellipse at the points *e′* and *f′*. Their horizontal projections are *ec* and *fc*. Draw the element CO of the construction cone as the second determining tangent line of the required plane. Pass the required plane T through the two intersecting lines OV and CO, each of which is tangent to the ellipsoid at the point O.

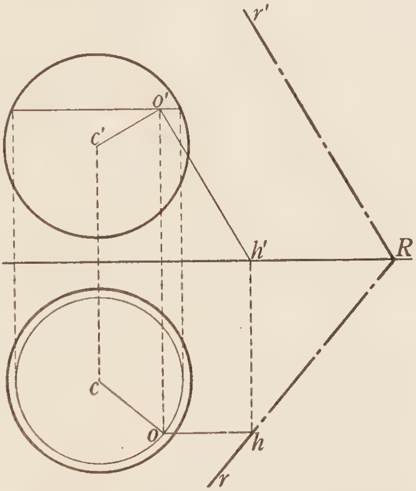


FIG. 168. Plane tangent to sphere through point on surface

Construction Two. Given the sphere whose center is C and the point O on its upper front portion, Fig. 168. Since the two tangent lines through the point O that determine the required plane are, in this case, both perpendicular to the radius of the sphere through O, the required plane must be perpendicular to the radius and can be located by the construction of Article 50. Draw the radius CO of the sphere. Pass the plane R through the point O and perpendicular to CO. Plane R is the required plane.

126. To pass a plane tangent to a sphere through an external point

Analysis. An infinite number of planes may be passed tangent to a sphere through an external point. Any plane satisfying the given conditions will be perpendicular to a radius of the sphere at the point of tangency and will also be tangent to a right circular construction cone that is drawn tangent to the sphere with its apex at the given point, and with the circle of tangency as the base.

Construction. Given the sphere whose center is C and the external point O, Fig. 169. Join the point O to the point C by the straight line OC. This line is the axis of a right circular construction cone tangent to the sphere. The required plane must be tangent to this cone; that is, contain an element of it. Set up the auxiliary coördinate plane 3 parallel to the axis OC by drawing *0–3* parallel to *oc*, and project both the sphere and the line OC upon it, as shown in the figure. The construction cone will appear on this plane as the isosceles triangle $o_3a_3b_3$. Set up the auxiliary plane 4 perpendicular to OC by drawing *3–4* perpendicular to o_3c_3 and project the base of the construction cone upon it as the circle whose center is m_4 and whose diameter is equal to a_3b_3. Select n_4 at any point on this circle as the plane-4 projection of a point N on the base of the cone, and return this point to H and V projections in the usual way. Draw the element ON of the cone and the radius CN of the sphere. Pass the plane S through the line ON and perpendicular to CN, by Article 50. Plane S is the required plane.

By choosing other points than N on the base of the construction cone, any number of planes may be drawn satisfying the given requirements. It should be noted that two H and two V projecting planes may be passed tangent to the sphere through the point O without the aid of a construction cone, as exemplified by the H projecting plane T in the figure; also that, if the point A or B had been chosen on the base of the cone in the plane-3 view as the point of tangency, the plane-4 view need not have been drawn.

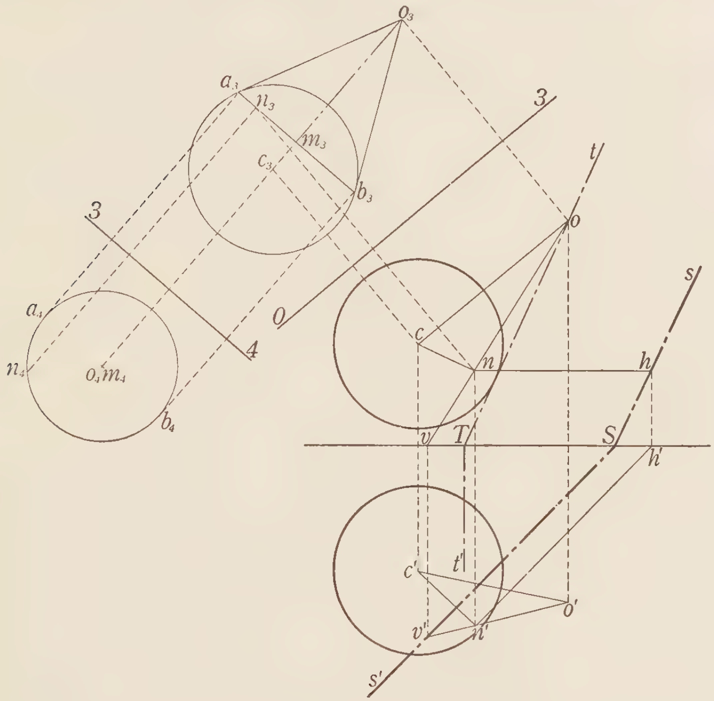


FIG. 169. Plane tangent to sphere through external point

127. To pass a plane tangent to any doubly-curved surface of revolution through an external point

Analysis. An infinite number of planes may be passed through an external point tangent to a doubly-curved surface of revolution. These planes will be tangent to a cone which has its apex at the given point and which is drawn tangent to the given surface. The cone may be circular, elliptical, right, oblique, or otherwise, depending upon the shape of the surface and the location of the point. No simple and direct method may be stated for locating this construction cone. Points on its base may be found by setting up auxiliary coördinate planes parallel to the axis of revolution, projecting the surface and point on each plane, passing two projecting planes through the point tangent to the surface in each new view, and returning each point of tangency thus found to H and V projections. Any plane-3 projecting plane so passed constitutes one solution of the given problem. Such a plane may be returned to H and V representations by a simple construction device, thus eliminating the need of finding the projections of the construction cone. If the required plane is restricted to one having a certain inclination to a coördinate plane, the cone must be obtained.

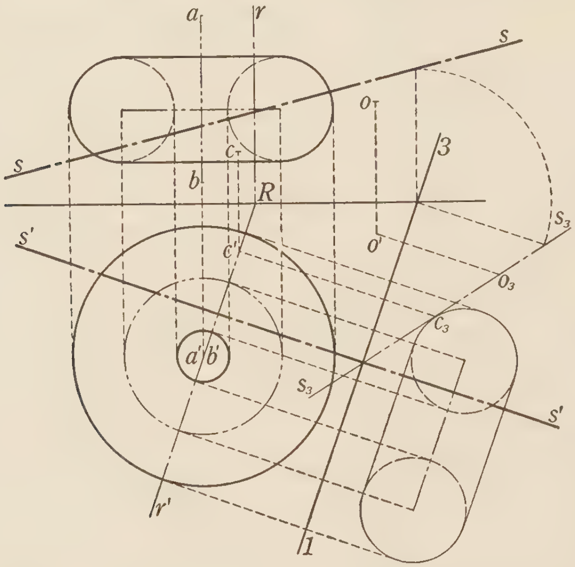


FIG. 170. Plane tangent to torus through external point

Construction One. Given the annular torus whose axis of revolution is AB, and the external point O, Fig. 170. Set up an auxiliary coördinate plane 3 parallel to the axis AB and project the torus and point upon it. The plane-3 view of the torus will be the same shape as the meridian section obtained by passing a plane R through the axis of the torus and parallel to plane 3. Draw s_3s_3 through o_3 tangent to the plane-3 projection of the torus at c_3. Plane S is perpendicular to plane 3; in other words, it is a plane-3 projecting plane. Therefore, $s's'$ must be perpendicular to the ground line *1–3*. The horizontal trace of the plane S is found by the usual return constructions. Plane S is one of the infinite number of planes that will satisfy the given requirements. By choosing other auxiliary coördinate planes parallel to AB, additional oblique planes satisfying the given conditions may be obtained.

It should be noted that two H and two V projecting planes may be passed externally tangent to the torus directly without the aid of any auxiliary planes. Further, when the point O is so located that a plane passed through it perpendicular to the axis of revolution cuts the given surface, as is the case in Fig. 170, the right-circular-cone method may be used, as in Construction Two, to obtain two of the possible solutions.

Construction Two. Given the oblate ellipsoid of revolution whose axis is AB, and the external point O, Fig. 171. Pass the plane R through the point O perpendicular to the axis AB. Plane R cuts a circle from the ellipsoid whose H projection is *cd*, whose V projection is the circle with center at f', and whose diameter is equal to *cd*. Draw the line OE tangent to this circle, and hence tangent to the ellipsoid at E. The line OE is one line in the required plane. Construct the right circular cone F tangent to the ellipsoid along the circle CED. The element FE of this cone is also a line in the required plane (see Article 125). The plane S, containing the lines OE and FE, is the required plane. A second plane can be obtained in this construction by drawing OE tangent to the ellipsoid on the upper sector of the circle.

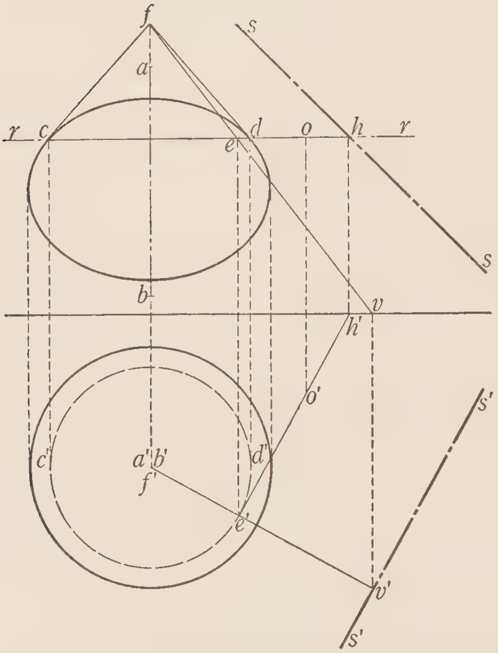


FIG. 171. Plane tangent to ellipsoid of revolution through external point

Discussion. If the given external point should be in the axis of the surface of revolution, all the planes satisfying the given conditions will be tangent to a right circular construction cone whose apex is at the given point and whose elements are tangent to the surface of revolution. The constructions will be similar to those of Fig. 171 except that the tangent OE may be drawn in any desired direction in the plane R.

128. To draw a line through a given point making a specified angle X with a given oblique plane

Analysis. If the given point is made the apex of a right circular cone whose axis is perpendicular to the given plane and whose base angle is equal to the given angle X, any element of the cone will satisfy the stated requirements.

Construction One. Given the point A, the oblique plane S, and the angle X = 60°, Fig. 172. Through the point A draw the line AC, of any length and perpendicular to the plane S, as the axis of the construction cone; $a'c'$ is then perpendicular to Ss', and ac is perpendicular to Ss. Set up an auxiliary coördinate plane 3 parallel to the line AC by drawing *0–3* parallel to ac, and project AC upon this plane at a_3c_3. The construction cone shows on plane 3 as the isosceles triangle $a_3b_3d_3$ with its base angle equal to 60°. Set up an auxiliary plane 4 perpendicular to the line AC by drawing *3–4* perpendicular to a_3c_3, and project AC upon this plane at a_4c_4. The construction cone shows on plane 4 as the circle whose center is a_4c_4 and whose diameter is equal to b_3d_3. Any element of the cone, such as AE, may be selected in the auxiliary views as the required line and returned to H and V projections in the usual way.

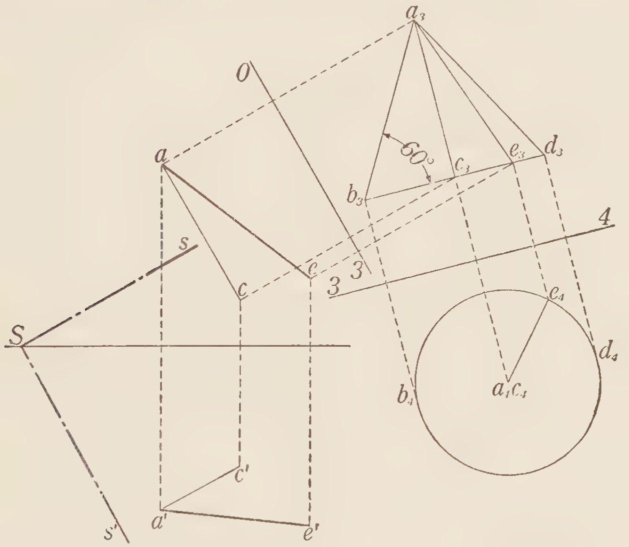


FIG. 172. Line through point at given angle with plane

Construction Two. Given the point A, the plane quadrilateral MNOP, and the angle X = 60°, Fig. 173. The axis of the construction cone cannot be drawn directly, as was done in Construction One, because the given plane is not represented by traces. Its direction may be determined, however, by constructing a line perpendicular to the surface MNOP at any point, as follows: Draw the H-parallel, BD, from one side of the quadrilateral to another. Through any point G on the line BD draw the V-parallel EF across the quadrilateral. Draw $g'k'$ perpendicular to $e'f'$ and gk perpendicular to bd. The line GK is perpendicular to both the line BD and the line EF (see Article 20) and hence is perpendicular to the quadrilateral MNOP. Through the point A draw the line AC parallel to the line GK as the axis of the construction cone. Proceed from this point in a manner identical to that of Construction One to obtain the projections of the line AL, one of the elements of the construction cone, and hence a line satisfying the given conditions.

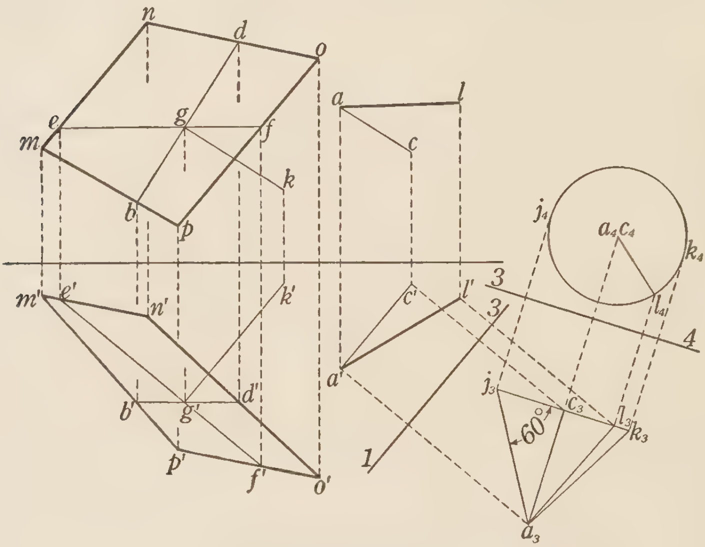


FIG. 173. Line through point at stated angle with plane

Discussion. It should be noted that the elements AB and AD of the cone in Fig. 172, and the elements AJ and AK of the cone in Fig. 173, could have been chosen from the plane-3 views as special solutions, in which case the plane-4 views need not have been drawn.

If it is desired for any purpose to have the base of the construction cone contained in the given plane, in either figure, the plane must be represented on the auxiliary plane 3 by a trace, s_3s_3, in the first construction, and by a straight-line projection, $m_3n_3o_3p_3$, in the second construction. The base of the triangular view of the cone on plane 3 must be made to coincide with this edgewise view of the given surface and then returned to H and V projections in the usual way.

The constructions of this article may be used for passing a plane through a point and making a specified angle with a given plane, since any plane passed tangent to the construction cone will satisfy the given requirement when the apex of the cone is placed at the given point. Any chosen element of the cone and the line drawn tangent to the base projection in the auxiliary view at the point where the element intersects this projection determine the required plane. The required plane can also be obtained by passing it perpendicular to an element of a construction cone that has been drawn with a base angle equal to 90° minus the given angle (see Article 116).

CHAPTER VII

PICTORIAL DRAWING

129. Pictorial drawings

Certain modifications in the space relations of the object, point of sight, projectors, and planes of projection, as maintained for orthographic projection, will alter materially the resultant appearance of the views on the principal coördinate planes, giving three distinct types of drawing or projection, called Isometric, Oblique, and Perspective. They are called pictorial drawings because of the similarity of their appearance to photographs and artists' sketches. They differ essentially from orthographic drawings in that only one view is required to convey the combined meaning of two or three orthographic views. Each is constructed by direct application of the principles of descriptive geometry, and although more difficult to construct than the familiar views of working drawings, they are all important in engineering and commercial work because of the ease with which they are read.

130. Isometric projection: definition

An *isometric projection* of an object is a view projected orthographically upon any plane, the object being so placed with respect to the plane that three of its mutually perpendicular plane faces are equally inclined to the plane. Therefore the only modification in the usual orthographic relations of object, point of sight, projectors, and plane of projection is the turning of the object into a very particular position with respect to the plane of projection and the restriction to one view. The point of sight, as in orthographic projection, is considered infinitely remote, and the projecting lines are perpendicular to the plane of projection.

It follows from the definition that the three lines of intersection of the three mutually perpendicular plane faces of the object are themselves equally inclined to the plane of projection and hence are proportionately foreshortened when projected upon it. That is, equal distances, measured on these lines or on lines parallel to them, are projected equally in the isometric drawing. It is this fact which gives the name "isometric" to this kind of projection, the root meaning of the word being "equal measurements."

131. Construction methods

The isometric projection of an object may be obtained in one of three ways. The advantages of each method of procedure appear in the following explanations of them.

Oblique-plane method. In Fig. 174 a cube, OABCM, is shown in the first quadrant with its faces parallel to the three coördinate planes. The oblique plane T, being equally inclined to the H, V, and P planes, is equally inclined to all the edges of the cube and so is properly placed to receive the required isometric projection of the cube. The isometric projection is found by drawing lines perpendicular to plane T from all the corners of the cube, locating the point in which each pierces the plane, and then connecting these piercing points, as shown. Because of the obliquity of the plane T with respect to the H, V, and P planes, the

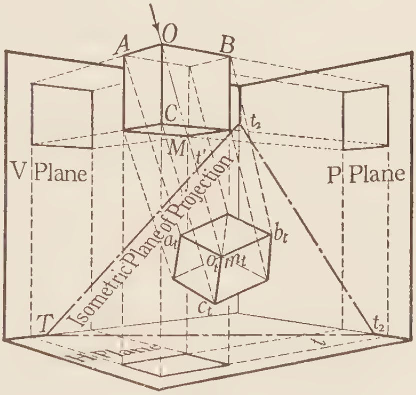


FIG. 174. Isometric projection by oblique-plane method

true size and shape of the isometric projection will not be seen from any of the usual orthographic view points until the plane has been revolved into one of the coördinate planes. In Fig. 175 is shown the usual orthographic representation of the same arrangement of cube, plane T, and projection of the cube upon T as was given pictorially in Fig. 174, together with the true shape of the isometric as found by revolving T into the vertical plane.

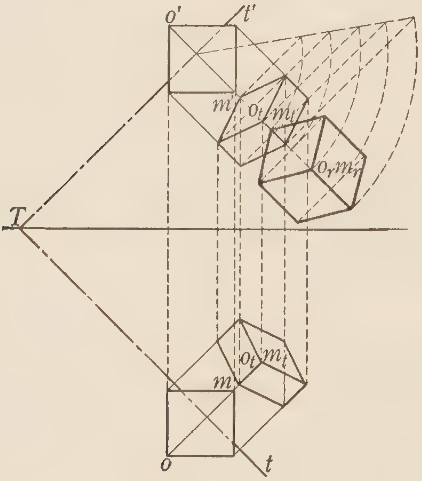


FIG. 175. Isometric plane revolved

Auxiliary-plane method. Since a solid diagonal of a cube is equally inclined to the three adjacent edges, the plane on which the isometric view is to be projected will, by the definition of an isometric projection, be perpendicular to this diagonal. In other words, the projection of a cube upon an auxiliary coördinate plane that has been set up perpendicular to one diagonal of the cube, and upon which this diagonal shows as a point, is an isometric of the cube (see Fig. 176). It will be observed that the resultant drawing is like that of Fig. 175, and, further, that the lines that are mutually perpendicular on the cube show at angles of 120° to each other on the isometric.

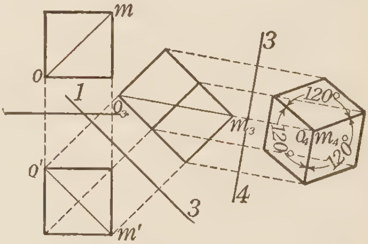


FIG. 176. Isometric by auxiliary-plane method

Special-placement method. The same result as was derived in Figs. 174, 175, and 176 may be obtained more directly by using one of the coördinate planes (for instance, the H plane in Fig. 177) as the plane of projection, and turning the cube to the proper position with respect to it; namely, with one solid diagonal perpendicular to this plane of projection, which is the relation between the object and plane in the three preceding figures. The procedure for constructing an isometric by this method is reducible to rule form, which will appear in the constructions of Article 135. This method or arrangement is the one generally employed in technical or commercial drafting. Its chief advantage is that for most objects the isometric can be drawn directly, without first constructing the top, front, or side views.

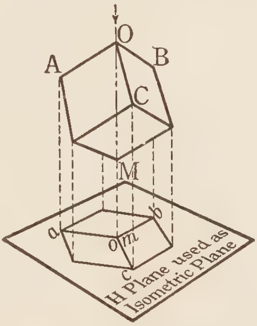


FIG. 177. Isometric on the H plane

The cube has been employed in the foregoing discussions because it lends itself easily to isometric representation. The three methods may be applied, however, in drawing an isometric of an object of any shape. It will aid in following the principles if a small cube is imagined to be an integral part of the object, say one corner, the entire object then being treated in conjunction with this cube, as the theory of the particular method may require.

132. The isometric plane

Any plane so related to an object that three chosen mutually perpendicular edges or lines of the object make equal angles with the plane, is called an isometric plane.

133. The isometric axes and isometric directrices

The three mutually perpendicular adjacent edges, OA, OB, and OC, of the cubes of Figs. 174 and 177, meeting at the extremity O of the diagonal OM, which is perpendicular to the isometric plane, are called the isometric axes of the object. Their projections, *oa*, *ob*, and *oc*, on the isometric plane are called the isometric directrices. In constructing the isometric projection of any object, three axes must be chosen perpendicular to each other and equally inclined to the isometric plane. The three directrices are inclined 120° to each other. Any lines of the object that are parallel to one or another of the axes chosen may themselves be called axes, and their projections upon the isometric plane are parallel to one or another of the directrices.

134. The isometric scale

In Article 130 it was pointed out that the three isometric axes, as well as all lines parallel to them, if of equal length, are foreshortened equally when projected upon the isometric plane. This foreshortening is best taken care of when measuring lengths on an isometric drawing by the use of an isometric scale. The isometric scale is one on which the unit of length is equal to the projected value of an inch, centimeter, or other standard unit that has been used in measuring along the isometric axes. The angle between the isometric axes and the isometric plane being about 35° 16′, the construction of Fig. 178 may be used in making the isometric scale. The difficulty of laying off the angle of 35° 16′ with accuracy makes this method less desirable than that of Fig. 179, which is easily constructed.

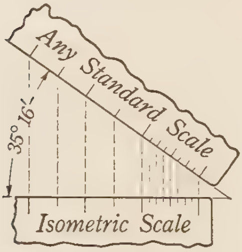


FIG. 178. Isometric scale

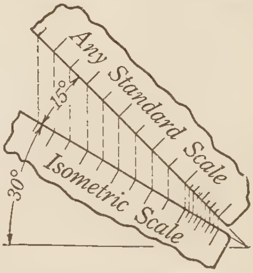


FIG. 179. Isometric scale

In practical usage the isometric scale is rarely applied, since the result would be merely the uniform reduction of all measurements of the drawing to about eighty-one hundredths of their true value, without affecting the shape of the drawing at all. If the drawing is to be dimensioned, as frequently is the case, it is imperative that a standard scale be used in order that the drawing may be scaled, if necessary, by the person reading it. In this connection it is emphasized that measurements on an isometric drawing may be made only along or parallel to the isometric directrices.

135. To construct the isometric of a given object

Let the given object be that represented by the two orthographic views of the first part of Fig. 180. Select any point *O* on the paper and draw the three directrices, *OE*, *OF*, and *OG*, inclined 120° to each other, to represent the three chosen axes, OA, OB, and OC, of the object. On these directrices measure from *O* the lengths of the corresponding edges of the object, as *OA*, *OB*, and *OC*, using either the isometric or a standard scale. Since all parallel lines of the object are projected as parallel lines in the isometric, the top face, AOCD, of the object will be represented in the isometric by the parallelogram *AOCD*, as shown in the figure. Other edges and faces are found in a similar manner.

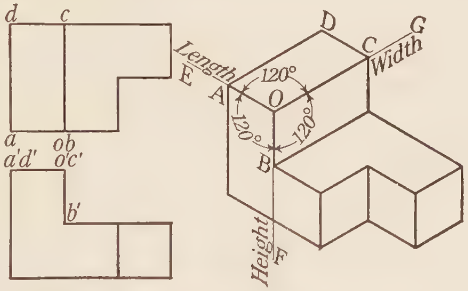


FIG. 180. Isometric construction from orthographic views

It is often advantageous to think of the three axes as representing the length, breadth, and thickness of the object; or the length, width, and height; or simply coördinate distances, *x*, *y*, and *z*; and to assign these names to the corresponding directrices. This will aid in distinguishing one from another and thus prevent measuring along the wrong directrix.

Edges of an object not parallel to an axis may be constructed in isometric only by locating two points thereon and connecting them. The points chosen are usually the extremities of the oblique lines, and these extreme ends are found in the isometric by determining their positions in construction lines drawn parallel to the axes. For example, in Fig. 181, the line BD of the object is oblique to the axes. The point B is in the construction line AC, as shown in the orthographic views, so that, in the isometric, *B* can be located by drawing *AC* in the usual way and laying off the distance *a′b′* or *c′b′* on *AC* in the proper direction. The point D is in the construction line CG. Its isometric projection is located on *CG* in the same way that *B* was located. Other points on the isometric are located in a similar manner.

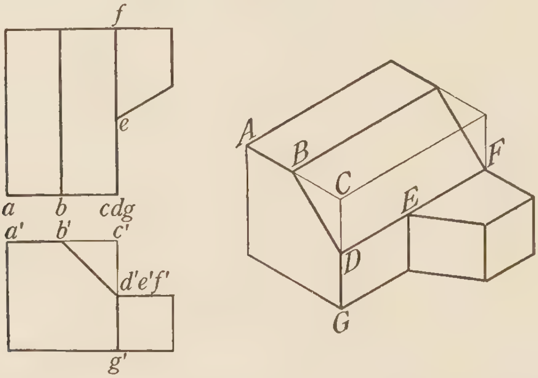


Fig. 181. Isometric construction from orthographic views

Any object may be projected isometrically, whatever its shape. If there are but three edges mutually perpendicular, as OA, OB, and OC, Fig. 182, these should be used as the axes, and all the irregular points, as D, E, and F, should be constructed by the coördinate method, as shown. If there are but two edges which meet at right angles to each other, the construction should be started, as shown in Fig. 183, with these edges projected as two directrices, *OA* and *OB*, and all other lines or vertices coördinated from these two axes. In general, it will be best to imagine the object inclosed in a rectangular parallelepipedon whose edges and faces coincide with, or contain, as many faces, edges, or vertices of the object as possible. Then, the isometric of the parallelepipedon having been constructed, the isometric of the object may be found by locating its vertices in the isometric of this framework. This method of construction was used in making the isometric of the hexagonal pyramid shown in Fig. 184.

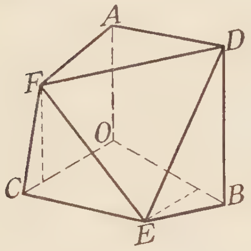


FIG. 182

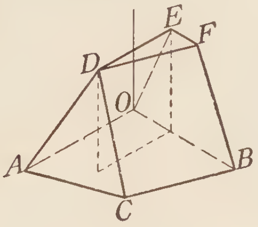


FIG. 183

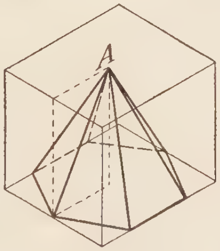


FIG. 184

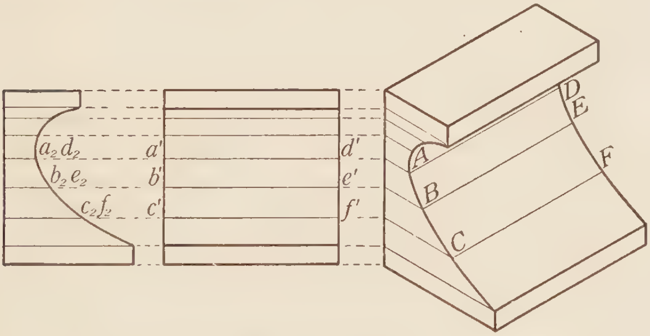


FIG. 185. Isometric construction of plane curve

The isometric of a plane curve is drawn by locating the isometrics of enough of its points to determine a smooth curve. Thus the isometric of the curved parts of Fig. 185 is determined by the points A, B, C, etc., whose coördinates are first measured in the profile view in the left portion of the figure and then transferred to the isometric. For a space curve, or curve of three dimensions, the process is similar, except that three coördinates must be measured for the determination of each point, and as many points must be located as are necessary to define the curve. Fig. 186 illustrates the method of constructing the isometric of a space curve from two orthographic views previously drawn.

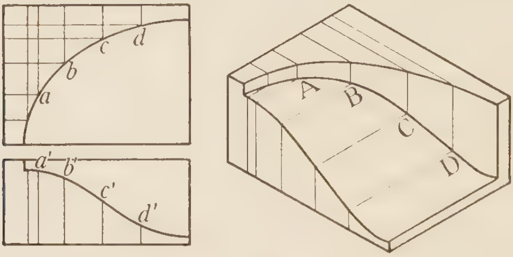


FIG. 186. Isometric construction of space curve

136. Oblique projection: definition

An *oblique projection* of an object is one obtained by means of projecting lines parallel to each other and inclined to the plane of projection. Only one plane is used, and the inclination of the projectors may be any angle desired.

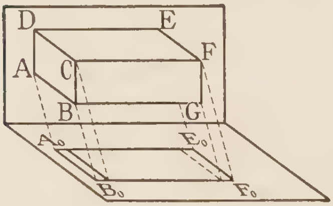


FIG. 187. Oblique projection

As in other forms of projection, the oblique projection of an object is determined by joining in the proper order the points in which projectors through the corners of the object pierce the projection plane. Fig. 187 is a pictorial representation of an object and the construction for finding its oblique projection upon the horizontal plane, the projectors, AA_0, BB_0, etc., being shown parallel to the vertical plane and inclined to the horizontal.

137. Placing the object

Since, by a system of parallel projectors, a plane figure is projected in true shape upon a plane to which it is parallel, the projectors being either perpendicular or inclined to the plane of projection, an oblique drawing is made by placing one of the faces of the object parallel to that plane. Further, for the same reason, the face chosen to be parallel is usually the one which contains the most circles, arcs, and other curves, or, if the object is composed of straight lines, the one which contains the most that are not parallel. These considerations contribute to facility of construction.

138. Inclination of projectors

The object having been properly placed with respect to the plane of projection, the direction of the projectors is then so chosen as to make visible in the projected view those faces of the object it is desired to show. In Fig. 187 the projectors are parallel to the vertical plane, as are two faces of the object. Consequently, in the oblique drawing only two faces, the top and left end, are seen, the front and rear faces projecting as two straight lines. This is not a suitable arrangement, since the oblique view thus obtained possesses little of the pictorial character desired. If the projectors are drawn in a direction inclined to the H, V, and P planes, as shown at AA_0, BB_0, etc., in Fig. 188, three faces of the object — the top, front, and left end — are visible, resulting in a view on the horizontal plane that more fully describes the shape of the object. These projectors could have sloped down to the left and rear, or down to the right or left toward the front, as well, the result being that other faces of the object would have been visible.

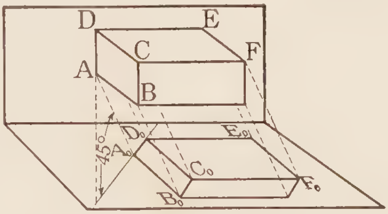


FIG. 188. Oblique projection

The shape of the oblique view will also vary with the degree of inclination of the projectors, the essential point of difference being the length of the projections of lines perpendicular to the plane of projection. The most generally used slope is 45°, whereby the projected length of each line perpendicular to the plane of projection is equal to the length of the line itself. This is the inclination used in Figs. 188 and 189.

139. Constructing the oblique projection

The method of applying the principles of descriptive geometry to the construction of an oblique projection is illustrated in Fig. 189. The object is placed with one face, ABCD, parallel to the plane to be used as the plane of projection (in this case the vertical); the two orthographic views, top and left side, are constructed in the usual way. Projectors, as EM, are drawn, at any desired inclination to the vertical plane (usually 45°), through the corners of the object, and the points, as *E*, *F*, etc., in which these projectors pierce the vertical plane are found. These points, properly connected, determine the required oblique projection.

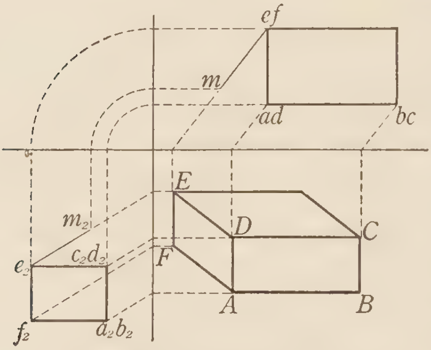


FIG. 189. Oblique projection from orthographic views

Since the faces parallel to the plane of projection are projected in true size and shape, an oblique drawing usually is constructed directly, without the use of the two orthographic views of Fig. 189. Thus, if the face *ABCD* is drawn in actual size or to a desired scale, from the known dimensions of the object, the parallel lines *AF* and *DE* can next be drawn, at any desired angle to *AD*, after which the remaining lines of the view can be added. If the projectors are inclined 45° to the plane of projection, the lengths of *AF* and *DE* are laid off full size, or to the same scale used for the measurements of *ABCD*. However, any scale, either larger or smaller than that used on the face *ABCD*, may be employed in measuring the third dimension of the object, since a line perpendicular to the plane of projection may be projected at any length whatever by suitably varying the inclination of the projectors.

By giving attention to the direction chosen for the receding axis, as the projection of an edge perpendicular to the plane of projection is called, the drawing may be made to show more fully the detailed shape of either the top or left side face of the object, as desired.

140. Lines neither parallel nor perpendicular to the plane of projection

The method of the last two paragraphs was employed in the construction of the oblique views of Fig. 190, the objects shown having all curved edges in planes parallel to the plane of projection. Obviously, many objects have curved or irregular edges that are not in parallel planes, in which case

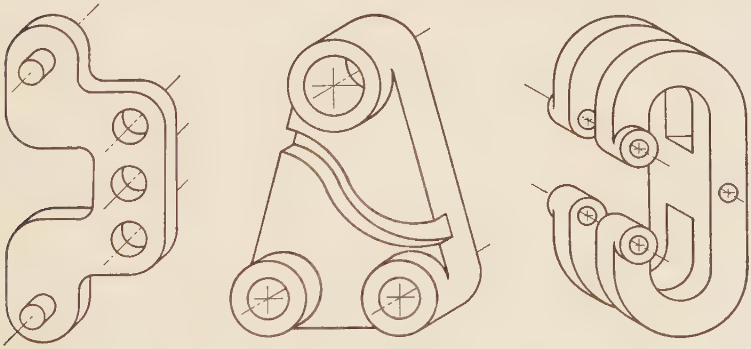


FIG. 190. Oblique projection of circles

the object is placed with as many curved edges as possible parallel to the plane of projection, after which the curves and irregular lines not parallel to that plane are determined point by point, in much the same way as are curves in isometric, that is, by coördinates. This method is illustrated in Fig. 191. A construction rectangle, ABCD, is circumscribed about the face of which the curve is a part, and coördinates with respect to the sides AB and AD are inserted for as many points, M, N, O, P, etc., as may be needed to determine a smooth curve. Then the oblique projection, *ABCD*, of this construction rectangle is drawn, in which are found the representations, *M*, *N*, *O*, *P*, etc., of the points chosen. A smooth curve through them is the required oblique projection of the curved line of the object.

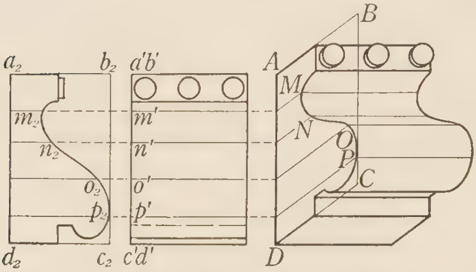


FIG. 191. Oblique projection of plane curves

An approximate method of obtaining the oblique projection of a circle by means of arcs of circles is shown in Fig. 192. The circumscribed square is constructed in oblique at *ABCD*, as before. Perpendicular bisectors of the sides are erected at *E*, *F*, *G*, and *H*, to intersect at the points *M*, *N*, *O*, and *P*, the centers of arcs which form the required oblique view of the circle. This method is applicable for any angle at the vertices *B* and *D* of the rhombus, but it should not be used for an angle of less than 45°, since the resulting curve does not approximate an ellipse. It cannot be used unless the receding axis *AB* is drawn to the same scale as *AD*.

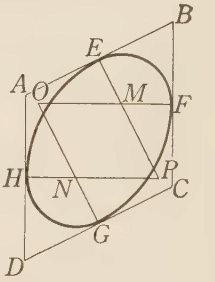


FIG. 192. Oblique projection of circle

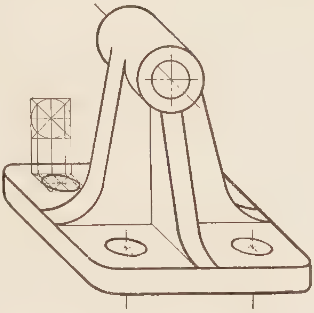


FIG. 193. Oblique projection of curves

If *AB* and *BC* are equal, and if the angle at *B* is 60°, the method of Fig. 192 may be used also as an approximate construction for the isometric of a circle. In this case the points *A* and *O* coincide and the points *P* and *C* coincide.

In Fig. 193 is shown the oblique representation of an object with curved edges in planes not parallel. While this is comparatively simple in shape, more complicated objects are treated in the same manner.

141. Applications of oblique projection

In addition to its application to formally constructed oblique drawings such as Figs. 190 and 193, the principle of this type of drawing is used occasionally for the purpose of rendering more legible the ordinary views of a machine drawing. This scheme, not extensively employed of late years, is the shading of some of the lines of an orthographic view in the manner shown in Fig. 194 in order to suggest the effect of three dimensions in each view which actually represents only two. The lines on the right side and lower

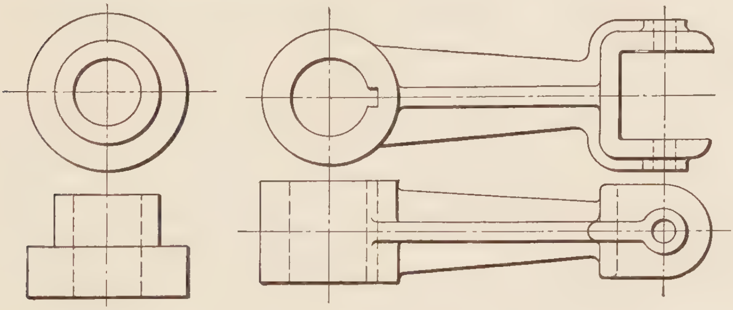


FIG. 194. Shade line drawing

edges of each view are made heavier than the others, as shown, thus giving the effect of thickness along a short receding axis. This is called shade line drawing, being in reality an oblique drawing in which the third dimension is only suggested.

Another application of oblique projection to practical drafting is that shown in Fig. 195, which is self-explanatory.

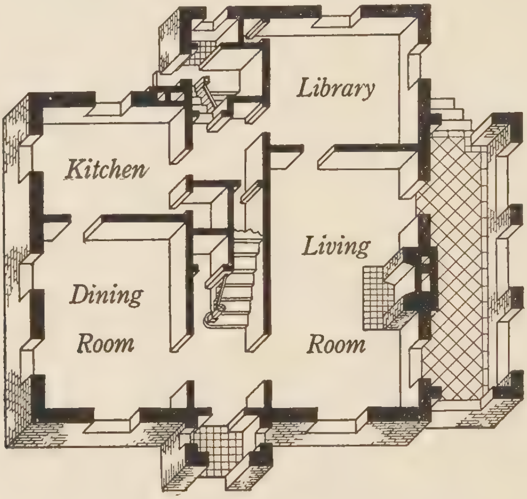


FIG. 195. Oblique drawing of floor plan

This use is generally limited to house plans made for display purposes, to be read by the very inexperienced layman, its advantage over the ordinary method of showing floor plans being apparent.

142. Comparison of various kinds of projection

Oblique and orthographic projections have the following principles in common: the point of sight is infinitely remote; the projectors are parallel to each other; and the object is placed, in general, with its most irregular plane face parallel to the plane of projection. They differ in that the projectors are inclined to the plane of projection in oblique drawing, and only one view is made, whereas in orthographic drawing the projectors are perpendicular to the planes of projection and any number of views may be used, the minimum being two.

Oblique and isometric projections are alike in that they have the point of sight infinitely remote, use only one plane of projection, and have parallel lines as projectors. Their essential points of difference are the obliquity of the projectors to the plane in the case of the former, and the obliquity of the faces of the object in the case of the latter.

143. Perspective drawing: definition

A *perspective drawing* is a representation, on a single plane surface, of an object as it would appear to the eye placed at a point comparatively close to the object. The projecting lines, or visual rays, as they are called, converge to the eye, thus being inclined at various angles to the surface on which the perspective is made. The representation is obtained by joining the points in which the visual rays pierce the plane, and presents to the eye exactly the same impression as the more distant object only when the eye is put at the same position with respect to the picture that it was imagined to occupy when the perspective was made. A perspective may be described as a likeness of an object represented on a transparent plane interposed between the object and the observer. In practical work, sheets of paper, canvas, blackboards, camera plates, and the like replace the transparent plane.

A perspective drawing is usually referred to as *the perspective*, or more simply as *the picture.* If properly constructed in line drawing and then softened with tone rendering, it should not be greatly unlike a good photograph of the object.

The artist draws his picture "by eye" and in so doing places each line by judgment based on the experience of previous study. The engineer's method of making a perspective may result in an effect no less artistic, and possesses the added merit of being reducible to rules which any draftsman may apply. A perspective drawing is constructed by joining, in proper order, the points in which the visual rays, or light rays, drawn from the significant points of the object to the position of the eye, called the point of sight, penetrate an imagined plane, called the picture plane, so set up as to cut all the visual rays. Consequently, the perspective of an object may be found by the rather simple process of locating the points in which a number of lines pierce the vertical plane, which is generally used as the picture plane. A perspective obtained in this way is called a mechanical perspective. Since the light rays, or projectors, converge, lines and angles are rarely shown in their true size, or plane figures in their true shape.

144. Steps in making a mechanical perspective

For the mechanical construction of a perspective a preliminary set-up must first be arranged, in sketch form or in the imagination only, in conformity with the required relations between the three main elements involved. The three main elements are the object, the point of sight, and the picture plane. In Fig. 196 is shown a sketch of such a set-up. The object is a segment of railroad track, and the requirements of this particular problem are that the picture plane shall be placed perpendicular to the track, and that the eye shall be located a specified distance in front of the picture plane and above the center line of the track. This step is merely preliminary to the actual construction of the perspective and is recommended because it aids the draftsman to make a conscious grouping of the factors used in his subsequent construction.

The next step is the construction of two or more orthographic views of the preliminary set-up, for which purpose the picture plane is made to coincide with the vertical plane, the object and point of sight being represented in their proper positions relative to the picture plane. The two orthographic views used may be a plan and front elevation, or a plan and a left or right side elevation. Fig. 197 shows a plan and left side elevation of the set-up of Fig. 196.

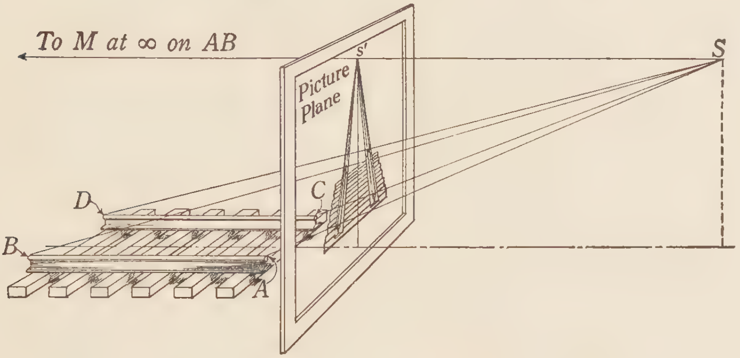


FIG. 196. Set-up for perspective drawing

The final step in drawing a perspective is the construction of lines, representing visual rays, from each point of the object it is desired to show in the picture, and the location of the points in which these rays pierce the picture plane. When enough piercing points are found, the perspective is completed by connecting them in the proper pattern for the object given. The way in which the perspective of an object is worked out from the orthographic views of the three elements concerned, as well as the way in which the principles of descriptive geometry enter into the construction, may best be illustrated by the

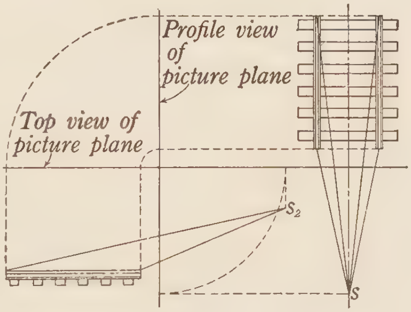


FIG. 197. Set-up for perspective drawing

detailed solution of a problem. It will develop in this solution that two distinct methods may be followed in obtaining the desired results. One is known as the Visual-Ray Method, the other as the Vanishing-Point Method. Both will be explained in succeeding articles.

145. Perspective by the visual-ray method

Let it be required to find the perspective of the object of Fig. 196. First are drawn a plan and a side elevation, or profile view, Fig. 198, of the three elements involved, using the set-up as a guide for fixing distances and positions. To simplify the drawing, only the rails of the track are lettered, — *ab* and *cd* in the plan, a_2b_2 and c_2d_2 in the profile. The picture plane is shown edgewise in both plan and profile views, as it was in Fig. 197. The eye is placed at the point S, which is represented by s, s', and s_2 in the drawing. These latter projections are determined in any stated problem by reference of the point S in feet or inches to some point on the object. Since the picture plane is projected as a line in both views, or since it is both an H and a P projecting plane, the piercing points, in the picture plane, of the rays drawn from A, B, C, and D to S are easily found. They are A_p, B_p, C_p, and D_p respectively. A_pB_p and C_pD_p are the desired perspectives of the rails. The method here used for the two lines is sufficient for the solution of problems involving any number of lines, that is, for any solid. It is

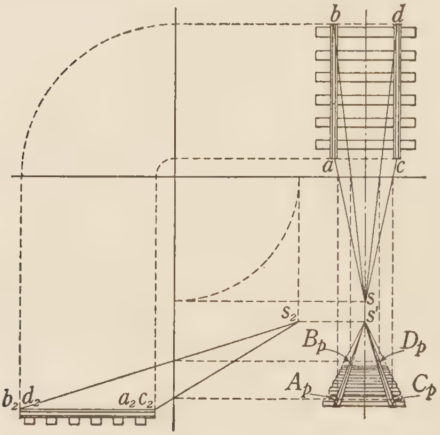


FIG. 198. Perspective by visual-ray method

called *the method of the visual ray* because all the points on the perspective are determined by use of visual rays. In Fig. 199 is shown a perspective obtained by the application of this method to an object resembling a building.

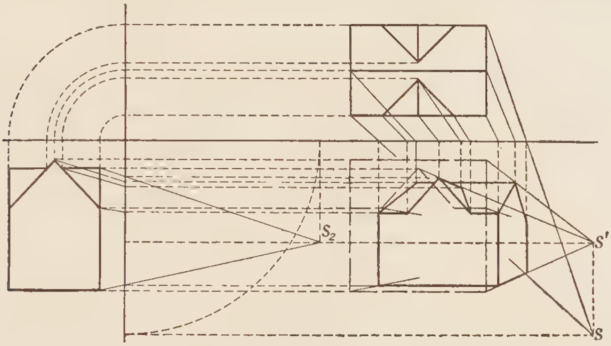


FIG. 199. Perspective by visual-ray method

146. Locating the object and the point of sight

The appearance of the perspective depends on the position of the point of sight with respect to the object and picture plane, or, if the eye and picture plane are considered fixed with respect to each other, on the way the object is turned with relation to the eye and picture plane. If one face of the object is against the picture plane, the perspective of that face coincides with the face itself. If one face is parallel to the picture plane, its perspective is similar in shape but reduced in size, depending on the distance of the face behind the picture plane. The object shown in both orthographic and perspective projection in Fig. 199 is placed with its front face parallel to the V plane, or picture plane, the perspective being constructed from the orthographic views by the visual-ray method. It will be noted that in perspective this front face, though smaller, is similar to the true shape of the front face of the object, as shown by the light dash lines in the vertical projection. Incidentally, Fig. 199 illustrates a reason for employing only the plan and a side elevation of the object when constructing a perspective, since the front elevation will almost invariably overlap the perspective.

147. Points in the picture plane

As stated in the preceding article, if a point, line, or plane figure is placed in the picture plane, it coincides with its own perspective and vertical projection. Use of this fact often shortens the work of finding the perspective of a line, for if the line can be made to pierce the picture plane conveniently, this piercing point is one point in the perspective of the line, after which only one other point in the perspective need be located. Thus, in Fig. 200, the line AB produced pierces the picture plane at the point P, whose perspective is P_p. The perspective B_p of the point B in the line AB is found by the visual-ray method. Then the line P_pB_p is the perspective of the entire line PB. If the perspective of the definite segment AB is desired, a construction for the point A may be added, similar to that for B.

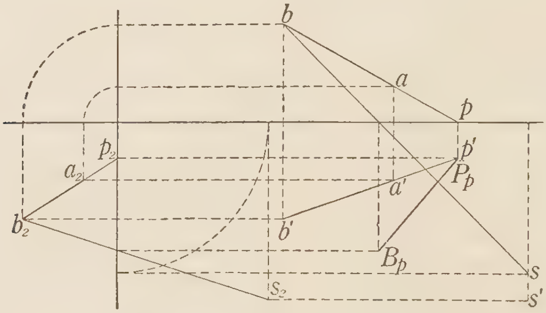


FIG. 200. Perspective of V piercing point of line

In the following article will be shown a method for finding another point which, when joined to the point P_p, determines the perspective of the infinite line of which AB is but a segment.

148. The vanishing point

The vanishing point in perspective drawing is the point in the picture plane at which is terminated the perspective of a given line of infinite length. It is that point in which a line, drawn through the point of sight and parallel to the given line, pierces the picture plane. It is the perspective of the infinite point on the given line. In Fig. 196 the vanishing point of the line AB is s'. Also, s' is the vanishing point of CD.

In Fig. 201 let it be required to locate the vanishing point of the given line AB. Through S draw the line SV, parallel to AB. Its projections, sv and $s'v'$, are parallel to ab and $a'b'$ respectively. The vertical projection, v', is the point in which the line SV pierces the picture plane; hence it is the required vanishing point of the line AB. It is the perspective of the infinite point on AB when the line is produced beyond B to infinity.

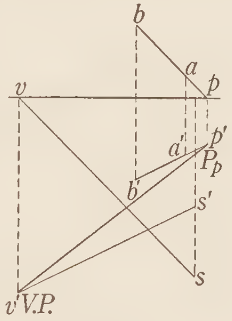


FIG. 201. Perspective of infinite point on a line

Further reference to Fig. 196 will aid in making clear why the foregoing construction gives the vanishing point. Imagine a point M chosen at an infinite distance on the line AB produced, and the visual ray drawn to the eye from this point. This ray is parallel to AB, and perpendicular to the picture plane, and the point s' in which it pierces the picture plane is the perspective of the point M. But no point can be taken in AB farther away than M, so that the perspective of AB extended infinitely terminates, or vanishes, at s'. A similar discussion applies to any line in space, whether it is perpendicular to the picture plane, as in Fig. 196, or inclined to it, as in Fig. 201.

Since the vanishing point, v', is one point in the perspective of the infinite line AB, Fig. 201, and P_p is a second, it will be seen that the line P_pv' is the perspective of AB from a point P, in the picture plane, to infinity. As in Article 147, if it is desired to lay off on $P_p v'$ the perspective of the definite segment AB, additional construction must be made, which will be illustrated later.

If two or more lines in space are parallel, they have a common vanishing point, for the line drawn through the point of sight parallel to one is parallel to all. This property makes the vanishing point a useful device in the construction of perspectives if the object is made up of several sets of parallel lines, since the perspectives of all the lines of any one set pass through their common vanishing point.

149. Perspective by the vanishing-point method

Let it be required to construct, by the method of vanishing points, the perspective of the object shown in Fig. 202. The object is to be turned with its longest edges at an angle of 30° to the picture plane to give a better view of it; the picture plane is to be *m* units in front of the upper front corner of the object; and the point of sight is to be *x* units to the left of the same corner, *y* units above, and *z* units in front of it.

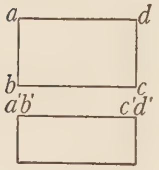


FIG. 202

Draw the plan *abcd*, Fig. 203, of the object in the desired position behind the picture plane, and locate the plan *s* of the point of sight at the proper position in front. Construct either a front or an end elevation of the object at any convenient position at one side of the sheet, as shown at $a'b'c'd'$, and locate the front view, s', of the point of sight with reference to this elevation and the point *s*. Now draw the lines SV and SV_1 through S, parallel, respectively, to AB and BC, by making *sv* parallel to *ab*, sv_1 parallel to *bc*, and $s'v'$ and $s'v_1'$ both parallel to $a'b'$, and find the points V and V_1 in which they pierce the picture plane. These points coincide with v' and v_1', respectively, and are the vanishing points for the lines AB and BC respectively, and for all edges of the object parallel to these lines (see Article 148). They are usually designated as V.P.L. and V.P.R., meaning the vanishing point to the left and right, respectively.

Next find the points P_p, P_{1p}, P_{2p}, and P_{3p} in the perspectives of the lines AD, BC, AB, and CD, respectively, by the theory of Article 147; then the lines P_pv_1', $P_{1p}v_1'$, $P_{2p}v'$, and $P_{3p}v'$ are the perspectives of the infinite lines of which AD, BC, AB, and CD are segments, respectively. Their intersections with each other at A_p, B_p, C_p, and D_p are the perspectives of the four points A, B, C, and D, as shown. The remaining three visible corners directly beneath A, B, and C are located in a similar manner to complete the required perspective of the object.

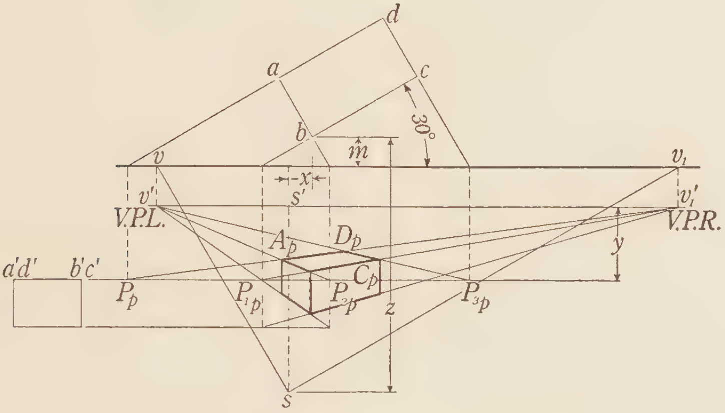


FIG. 203. Perspective by vanishing-point method

It will be noted in Fig. 203 that the view $a'b'c'd'$ is like the vertical projection of the object when one end is parallel to the V plane, and not an actual profile or vertical plane view projected from the plan *abcd*, as was the second view of the object in Fig. 199. This is because it is used in the vanishing-point method for determining vertical measurements only, or heights, in the perspective, as the figure shows, and for this reason may be any elevation.

150. Comparison of the vanishing-point and visual-ray methods

The vanishing-point method is generally superior to that of the visual ray because it may be used if one has a plan and any elevation, as a set of blue-print drawings, of a building or other object whose perspective is desired (see Article 149). The actual construction of the perspective can begin as soon as the plan has been arranged in the proper relation to the stipulated point of sight — say, as shown at *abcd*, Fig. 203 — and the sheet containing an elevation has been placed at any convenient position at one side, as *a'b'c'd'*. It is necessary that the two views be drawn to the same scale. The elevation may be front or rear, or right or left side, since it is used only for the measurement of heights, whereas, if the visual-ray method is employed, the elevation must be an actual projection of the object upon the end plane (see Article 145).

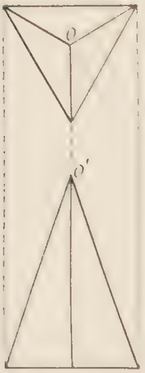


FIG. 204

A further advantage is the rapid determination of one point (the vanishing point) common to all lines of a parallel set, which not only reduces the amount of construction but at the same time furnishes an aid to accurate drawing.

The construction of the perspective of an object which does not comprise sets of parallel edges, as, for instance, the pyramid of Fig. 204, which has no two lines parallel, is not expedited greatly by the vanishing-point method, so that quite often a combination of this and the visual-ray method proves most effective. The minimum construction is shown in Fig. 205 for the combined methods. The two perspectives C_px' and C_px_1' of the two base lines, CB and CA produced to infinity, are located by means of their vanishing points, after which the definite points A_p and B_p are found on these edges by drawing the H projections of visual rays. The perspective O_p of the apex is located, by

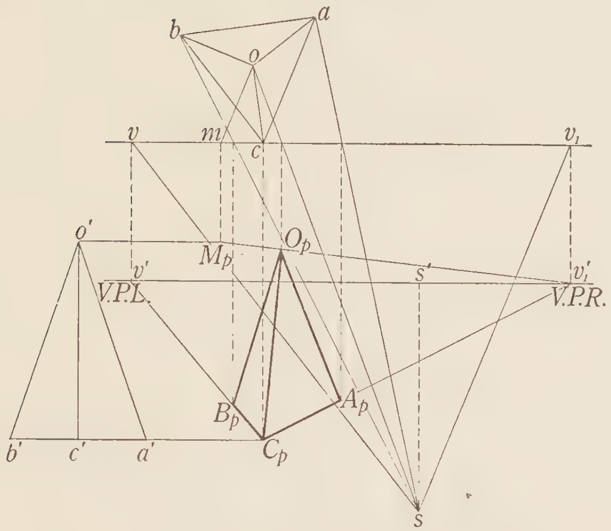


FIG. 205. Perspective by combined methods

means of a visual ray, on the perspective of the line OM passed through O, after the perspective of the infinite line OM has been determined by use of its vanishing point.

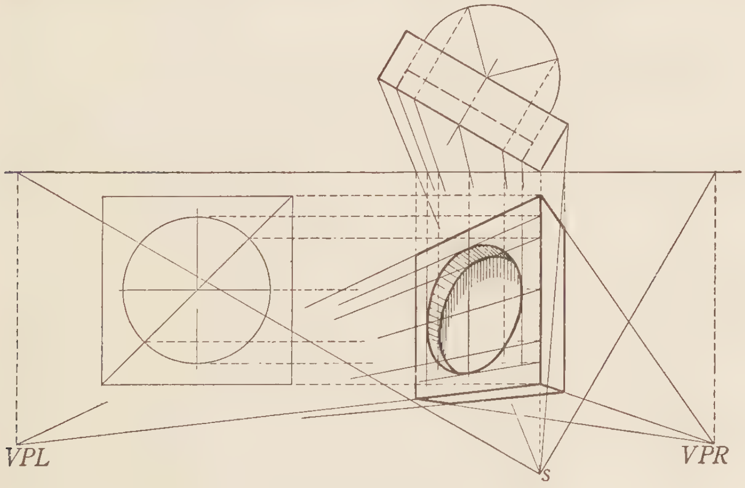


FIG. 206. Perspective of curved lines

151. Perspective of curved lines

The perspective of a curved line is determined by locating the perspectives of several points in it. This is done in much the same way as that used for the point O in the preceding article, and illustrated in Fig. 206, which shows the method applied to the perspective of a circle whose plane is perpendicular to the horizontal plane and inclined to the vertical plane. The perspective of a three-dimensional curve is found in a similar manner.

152. Shadows, shades, and shade lines: definitions

A *shadow* is that part of the surface of an illuminated object that is devoid of direct rays of light because of the interposition of some opaque body between it and the source. In a physical sense, a shadow is a volume and comprises all that space from which the light is excluded by any opaque body. In architectural work, however, in which it recurs, the term "shadow" has reference to a limited area of unlighted surface.

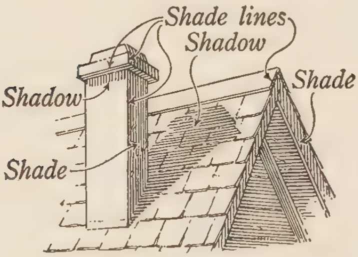


FIG. 207. Shadows, shades, and shade lines

A *shade* is that portion of the surface of an object that is turned away from the light. The distinction between shadow and shade is sometimes difficult to make. Indeed, occasionally the one will merge into the other in such a way that no line of demarcation can be drawn. In general, if an object "stands in its own light" its unlighted part is in *shade.* If some other object "stands in its light," the unlighted part is in *shadow.*

In Fig. 207 the shadow of the chimney is shown upon the roof, and the right face of the chimney is in shade. Also, the right end of the building is in shade, while the shadow of the building would be found upon the ground, or on neighboring trees or structures. The coping on top of the chimney casts a shadow on the front face of the chimney as well as on the roof, its under side and right face being in shade.

The *shade line,* or line of shade, is the line which separates the lighted area of an object from the part in shade. For example, the right vertical edge of the front face of the chimney, Fig. 207, up as high as that face is lighted, is one of the shade lines of the object. The shade line may also be defined as the line of contact with the object of a cylinder or prism of light rays which envelops the object. It may be a straight line, such as one of the edges of the object, as shown in Fig. 207, or it may be curved, as the great circle indicated on the sphere of Fig. 208. In this latter case, it is the line of intersection of the sphere by a plane through its center perpendicular to the light rays. The shadow cast by a shade line forms the outline of the shadow of the given object.

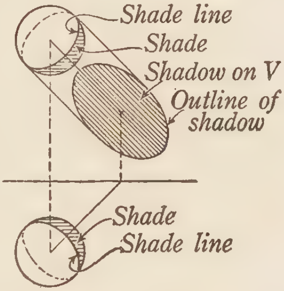


FIG. 208. Shadow, shade, and shade lines

Shadows and shades are represented in architectural drawing for the purpose of enriching the views of a structure and giving them fuller meaning. Tonal rendering adds relief to an otherwise flat elevation, resulting in a feeling, as experienced by the observer, that in a measure approaches that produced by the object itself. Washes, laid over with a brush, and shading applied with soft pencil are graduated by the skilled artist to suggest the variations of light and shade in nature.

Before these are applied it is necessary to work out the shape of the area to be covered. This must be done, when either the object casting the shadow or the one on which the shadow falls is irregular in shape, by tedious point-by-point construction, if accuracy is desired, in much the same manner as are lines of intersection of curved surfaces. Lines are drawn to represent rays of light through the various points which cast shadows, their piercing points with floor, wall, column, pilaster, molding, or other solids being the shadows of the points on these objects. When enough points have been located, the outlines of the required shadows are determined by connecting them in the proper order.

First-quadrant projection is employed in all the illustrations on shadow construction, in conformity with the prevailing custom in architectural drawing.

153. Conventional light ray

The light rays in architectural work are assumed to be parallel, and to extend downward to the right and rear in the direction of one diagonal of a cube whose faces are parallel to the coördinate planes. This means that their projections are drawn at an angle of 45° to the ground lines, and that the slope of the ray itself with respect to the H, V, and P planes is about 35° 16′, as shown in Fig. 209. The shadow of any given point is found wherever the ray through the point pierces the first object or surface encountered. Thus the shadow of the point A, Fig. 210, is on the V plane at a_s'. The shadow of the point B falls at b_s on the H plane, and that of C, at c_{s_2} on the P plane. In representing a shadow on the H, V, and P planes, and on those plane faces

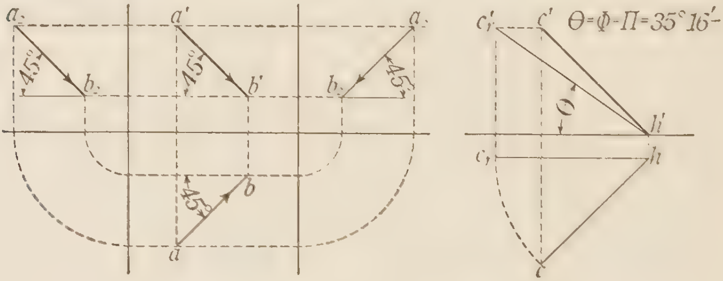


FIG. 209. Direction of light rays

of any object that are parallel to them, the projections of the shadow that fall in the ground lines are left unmarked (see Fig. 210). Except for purposes of discussion, even the vertical projections need no notation, since, in general, the area of a shade or shadow is made noticeable by rulings, as in Fig. 208. In marking out the extent of a shadow greater difficulty is experienced in determining which points or lines cast the shadow than in locating the points in which the rays pierce one surface or another.

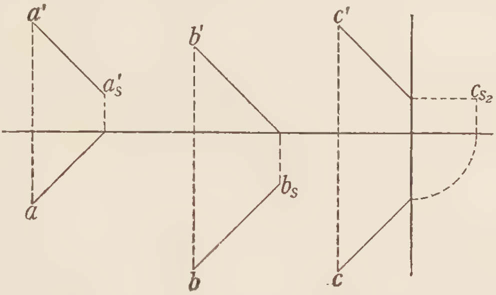


FIG. 210. Shadows on coördinate planes

154. Point-by-point method of drawing shadows

Let it be required to find the shadow of the curved line AB in the first drawing of Fig. 211. Rays are drawn through various points A, E, etc., of the line AB, their V piercing points, a_s', e_s', etc., being joined to give the required shadow, $a_s'b_s'$. Seven points are shown in the construction, three of which, a_s', e_s', and b_s', are lettered. In the second drawing of Fig. 211 the shadow $c_s'e_s'd_s'$ cast upon the short piece of molding MNO by the curved line CD near it, is constructed in a similar manner. Since the surface upon which the shadow falls is partly curved, a larger number of rays than for the line AB must be employed to obtain accurate results.

In Fig. 212 the conditions are the same as in Fig. 211 except that the lines whose shadows are to be found are straight lines. The segment $e_s'b_s'$ of the shadow which falls on the cylindrical part NO of the molding, in the first drawing of Fig. 212, is determined point by point, as before. While this method may be followed also for the segment

$a_s'e_s'$ which falls on the plane face MN, use should be made of the fact that a straight line casts a straight shadow upon a plane. Therefore only two rays are needed. The first is the one through A, locating the shadow of A at a_s'. The second should be the ray through the point in AB whose shadow falls on the line of intersection, NN, of the plane and cylindrical faces of the molding. To find this point, draw

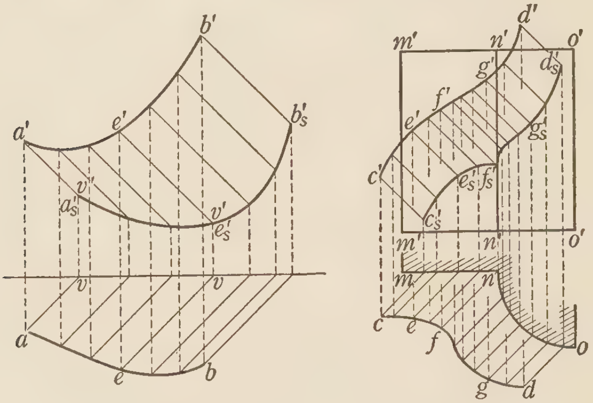


FIG. 211. Shadows of curved lines

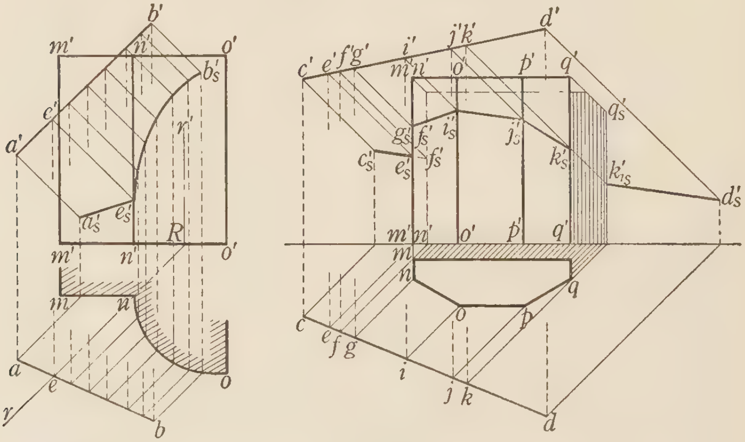


FIG. 212. Shadows of straight line on plane and curved walls

the H trace, *Rr*, of the plane of rays through NN. The point E in which AB pierces this plane is the point desired, its shadow being found at e_s', the straight segment of shadow then being drawn by connecting a_s' and e_s'.

This construction for finding the shadow of the point E in the line AB is employed in the second drawing of Fig. 212 to determine the points g_s', i_s', etc., at which the shadow of the line CD breaks around the vertical edges NN, OO, etc., of the prism MQ, the total shadow, $c_s'e_s'g_s' \cdots d_s'$, being partly on the V plane and partly on the prism. The ray through the point K touches the prism at k_s', and continues until it pierces the V plane at k_{1_s}', so that, in a sense, the point K has two shadows. The point k_s' is sometimes called a point of loss.

Since the face OPPO is parallel to the V plane, the segment $i_s'j_s'$ of the shadow on it is parallel to $c_s'd_s'$.

In the second part of Fig. 212, in addition to the shadow of the line CD upon the prism and upon the V plane, the shadow of the prism itself is shown, partly on the H plane and partly on the V plane. Most of the shadow on the V plane falls directly behind the prism and so is invisible in the front view. The outline of this hidden area is indicated for the purpose of making the drawing complete, although ordinarily it would be omitted altogether.

155. Principles

Several fundamental principles may be stated which aid in simplifying the constructions necessary in drawing shadows.

1. If a line is parallel to a plane, its shadow on that plane is parallel, and equal in length, to the line itself. Fig. 213 shows the shadows, on the V plane, of three straight lines, AB, CD, and EF, and of a circle with center at C, all parallel to the V plane.

2. If a line is perpendicular to a coördinate plane, its shadow on that plane is a straight line inclined 45° to the ground line (see Fig. 214).

In both numbers 1 and 2 the shadows of the lines shown may be considered as traces, on the H or V plane, of planes of rays passed through the given lines (see Fig. 215). In some cases it may prove advantageous to think of the shadow as being cast by a cylinder of rays; for example, in finding the shadow of the circle in Fig. 213.

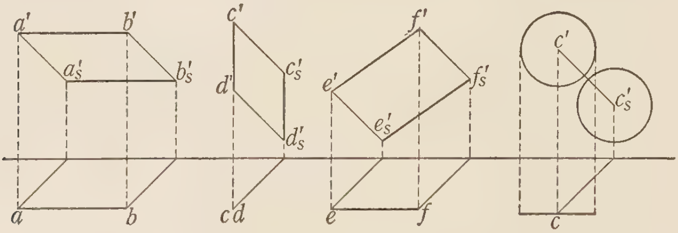


FIG. 213. Shadows of lines on coördinate planes

3. If the shadow of a straight line falls partly on the H plane and partly on the V plane, it breaks at the ground line in such a way that the part on each is governed by numbers 1 and 2 above. In Fig. 215 are shown three possible cases. Note here, again, that the shadows on the H and V planes are definite parts of the traces of a plane of rays.

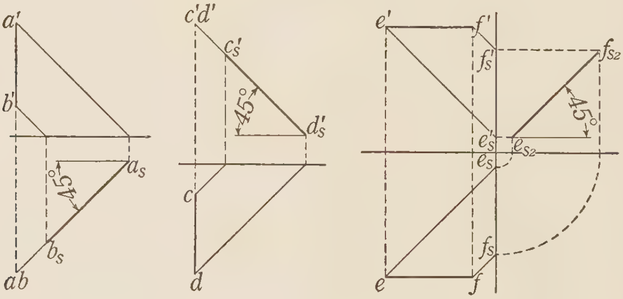


FIG. 214. Shadows of lines on coördinate planes

4. If a line AB parallel to the H and V planes casts a shadow on a vertical broken wall, column, or molding, as in the first drawing of Fig. 216, the shadow has the same shape

as the inverted horizontal projection of a right section of the wall, column, or molding. Application of this principle makes it necessary to locate the shadow of but one point, as a_s', after which the remaining parts may be drawn in directly.

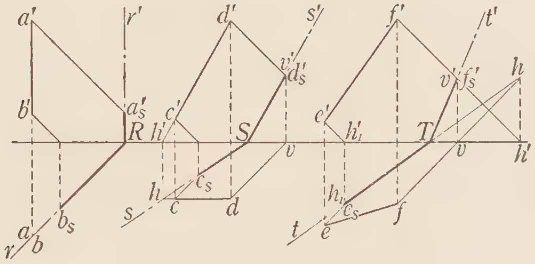


FIG. 215. Planes of rays

5. If a vertical line CD casts a shadow upon a molding parallel to the H and V planes, as in the second drawing of Fig. 216, the shadow has the same shape as the reverse of a profile plane section.

6. If a circle is parallel to the H plane, its shadow on the V plane is an ellipse which is determined as shown in the first drawing of Fig. 217. The shadow, $d'e'f'g'$, of a square circumscribed about the circle is first constructed, and the midpoints, *1*, *3*, *5*, and *7*, of its sides are determined. The distance $e'v'$ is laid off on $a'v'$ to give o and p. Lines through

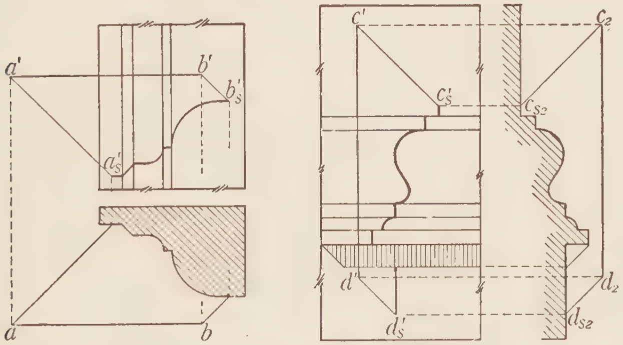


FIG. 216. Shadows of straight lines on plane and curved walls

o and *p* parallel to *d′e′* cut the diagonals of *d′e′f′g′* in the points *2*, *4*, *6*, and *8*. The points *1* to *8* determine the shadow.

7. If a circle is parallel to the P plane, its shadow on the V plane is an ellipse which is determined as shown in the second drawing of Fig. 217.

8. The shadow cast upon the H or V plane by a sphere is an ellipse whose center and major and minor axes are determined as shown in Fig. 218.

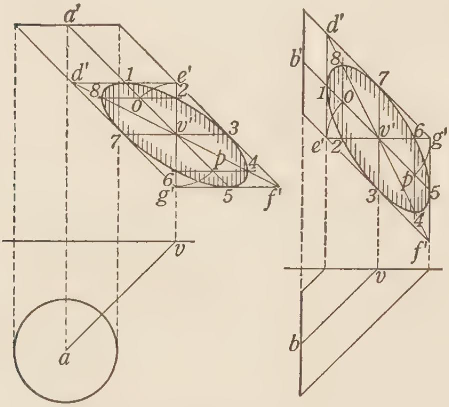


FIG. 217. Shadows of circles on coördinate planes

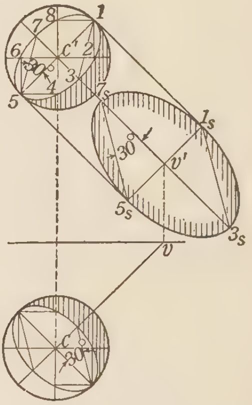


FIG. 218. Shadow of sphere on coördinate plane

156. Shade line of a cylinder

A circular cylinder perpendicular to the H plane has half its surface lighted, but not the half that is visible in the V projection. The shade lines are the elements of tangency, AB and CD, Fig. 219, of planes of rays. Since AB is on the invisible half of the cylinder, it is not considered or shown. That part of the elevation to the right of *c′d′* is usually shaded as in the figure.

157. Shade line of a cone

The shade lines of a cone whose axis is perpendicular to the H plane are obtained as shown in Fig. 219. They are the elements of tangency, AB and AC, of planes of rays. It will be noticed that more than half of the curved surface is lighted. The amount of shaded area of a cone with its axis perpendicular to the H plane decreases as the altitude decreases, until the base angle becomes equal to the slope of the light rays, when all the surface is lighted. If the cone is inverted, the areas of light and shade are interchanged, and unless the base angle is equal to the slope of the light rays or greater, all the surface is in shade.

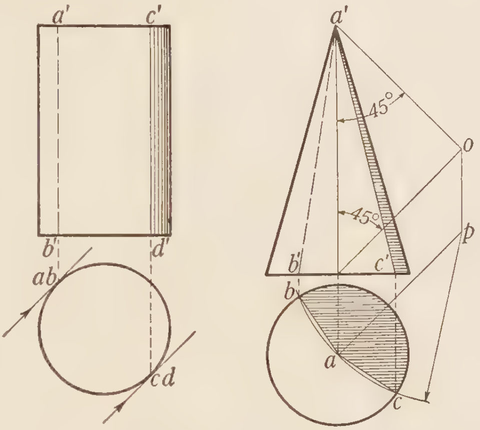


FIG. 219. Shade lines of cylinder and cone

158. Shade line of a sphere

The cylinder of rays tangent to a sphere touches it along a great circle which is the line of intersection of the sphere and a plane drawn through the center and perpendicular to the light rays. This circle is the line of shade, its projections being found by the construction of Fig. 218.

All the constructions of Figs. 213 to 219, inclusive, are based on rigid applications of descriptive geometry principles, and are given in the manner shown because they may be drawn quickly and accurately by these short-cut methods.

The routine construction required in the determining of shadows may be greatly reduced by the application of the foregoing principles. In fact, the skilled architectural draftsman, having mastered them by study and experience, is generally able to sketch in the shadows on an elevation almost entirely by eye, and with comparative exactness.

PROBLEMS

The problems of the following pages are grouped under headings corresponding in meaning to those used in the main section of the text for the articles, indicating thereby the principles that the problems are to exemplify. The problems and figures are numbered consecutively, though not in agreement in any individual case. References to article, figure, and problem numbers are inclusive in every instance.

Problem specifications are of two types: first, those referring to a group of figures on one page, called a "drill sheet," and, second, those referring to a single figure which appears on the same page with, or a page contiguous to, the problem specification. The first nineteen pages of figures constitute the drill sheets. Except in a few instances the problems referring to these sheets are given first under each article heading. The instructor will assign problems of this type for class and home drill by stating the problem and figure number, which he may do in any case without reference to the drill sheets themselves, thus: Problem 12, Fig. 31, or Problem 12, Fig. 52. The page number may be added if desired.

The figures should be enlarged to at least twice their present size in reproducing them on class plates. With a two-to-one enlargement the solutions will remain within a $3'' \times 3''$ to $4'' \times 4''$ space, thus making it possible to place from six to eight problems on a $9'' \times 12''$ sheet without crowding. Reproduction of the drawing by eye will suffice in most cases, but where the relation of parts must be maintained accurately, vertical and horizontal scales have been added to the drawings,—the former on either or both sides of the sheet, the latter along the ground lines. The unit of measurement marked on the scales is one eighth of an inch. Points falling between scale points are to be considered as on the sixteenth-inch mark.

Lines that are drawn parallel or perpendicular to other lines are to be detected by inspection of the drawing only. No closely parallel or perpendicular lines are shown on the drawings, as a rule. Traces of planes are inclined to the ground lines at angles of 15° and multiples thereof.

Unless specifically stated otherwise, the solution of any problem comprises the H and V projections or views, although constructions on other planes may be required in obtaining these views.

Where two or more magnitudes, such as a point and a line or a line and a plane, for instance, are represented on the drill sheets, they are referred to in the problem specifications as a "set," for want of a better term, and their reproduction on the student's drawing sheet, with indicated relations maintained to scale, is known as the "set-up," as is the imagined arrangement of the magnitudes in space.

Where distances are specified in the problems in inches, full scale is intended irrespective of the enlargement used. Where distances are indicated by a graphical means, the indicator should be enlarged in proportion to other parts of the drawing. The bracketed coördinates of points are given in inches.

A knowledge of the principles and conventions of elementary mechanical drawing is assumed. It is also assumed that the student knows at least one mechanical method for constructing an ellipse and other simple plane curves.

In obtaining the solution of any of the following problems, the student should adhere to the scheme employed throughout the text, which is stated in summary form on page 50. The analysis of an involved problem should be written out before the constructions are started. It should contain not only statements of the main elements of attack to be used in developing the necessary constructions but also any facts of a restrictive character which might control or limit the method and number of solutions. The constructions should be executed neatly and accurately. The lettering of each projection or trace as it is drawn is imperative, except where the same construction is repeated a number of times in a single problem.

DRILL SHEET NUMBER 1

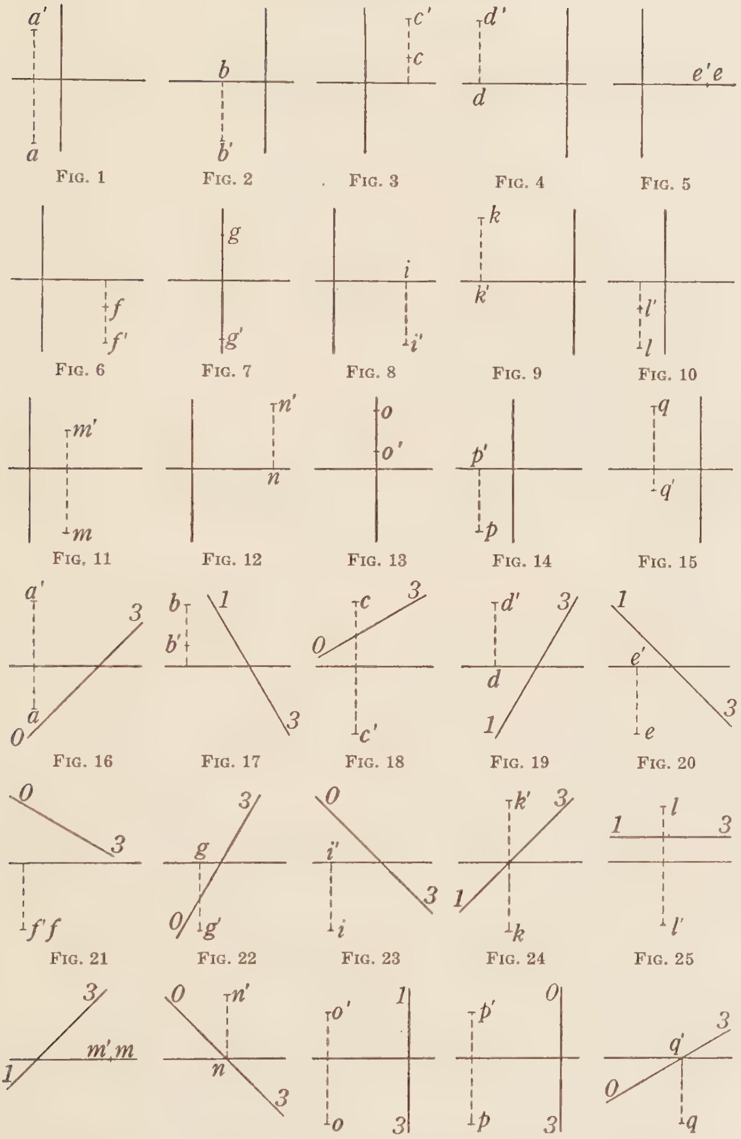


FIG. 1 FIG. 2 FIG. 3 FIG. 4 FIG. 5

FIG. 6 FIG. 7 FIG. 8 FIG. 9 FIG. 10

FIG. 11 FIG. 12 FIG. 13 FIG. 14 FIG. 15

FIG. 16 FIG. 17 FIG. 18 FIG. 19 FIG. 20

FIG. 21 FIG. 22 FIG. 23 FIG. 24 FIG. 25

FIG. 26 FIG. 27 FIG. 28 FIG. 29 FIG. 30

FIGS. 1–30. Point with profile or auxiliary coördinate plane

DRILL SHEET NUMBER 2

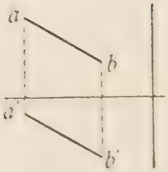


FIG. 31

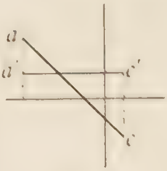


FIG. 32

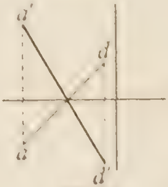


FIG. 33

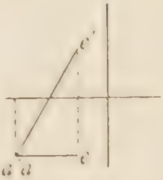


FIG. 34

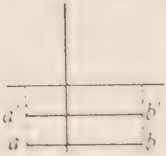


FIG. 35

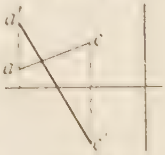


FIG. 36

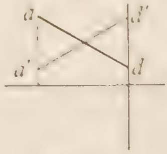


FIG. 37

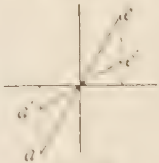


FIG. 38

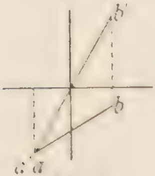


FIG. 39

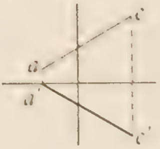


FIG. 40

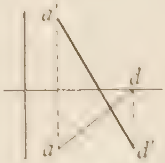


FIG. 41

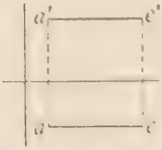


FIG. 42

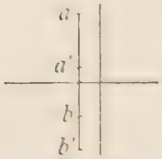


FIG. 43

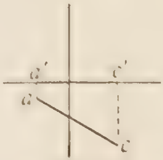


FIG. 44

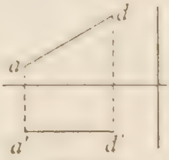


FIG. 45

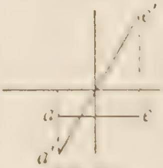


FIG. 46

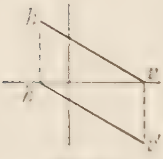


FIG. 47

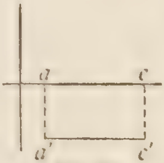


FIG. 48

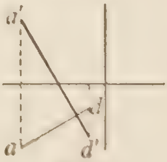


FIG. 49

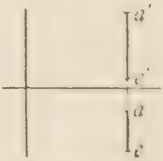


FIG. 50

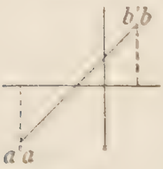


FIG. 51

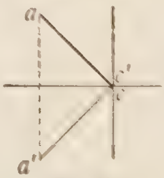


FIG. 52

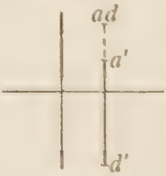


FIG. 53

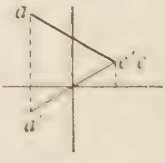


FIG. 54

FIGS. 31–54. Line with profile coördinate plane

DRILL SHEET NUMBER 3

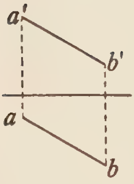


FIG. 55

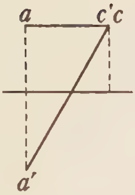


FIG. 56

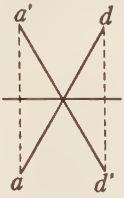


FIG. 57

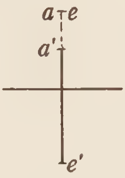


FIG. 58

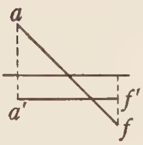


FIG. 59

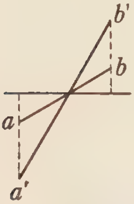


FIG. 60

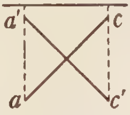


FIG. 61

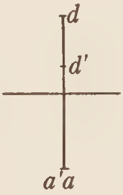


FIG. 62

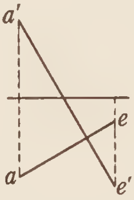


FIG. 63

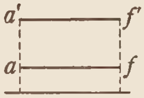


FIG. 64

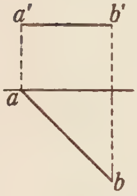


FIG. 65

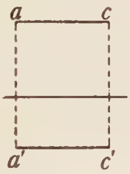


FIG. 66

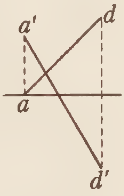


FIG. 67

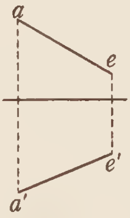


FIG. 68

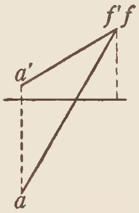


FIG. 69

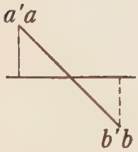


FIG. 70

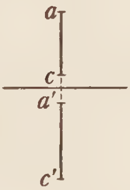


FIG. 71

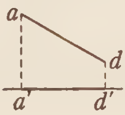


FIG. 72

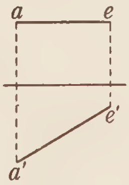


FIG. 73

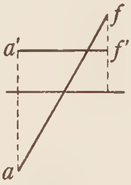


FIG. 74

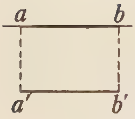


FIG. 75

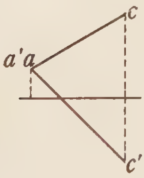


FIG. 76

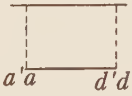


FIG. 77

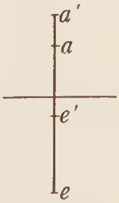


FIG. 78

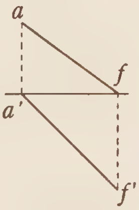


FIG. 79

FIGS. 55–79. Line in space

DRILL SHEET NUMBER 4

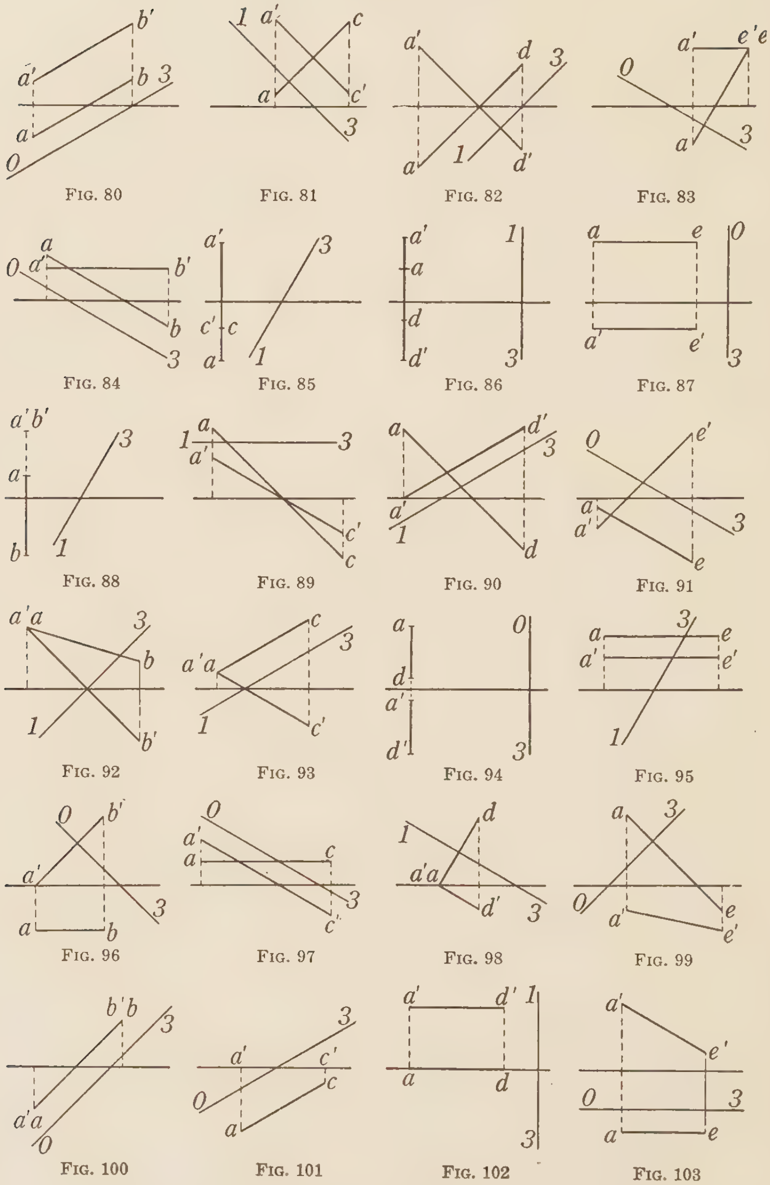


FIG. 80 FIG. 81 FIG. 82 FIG. 83

FIG. 84 FIG. 85 FIG. 86 FIG. 87

FIG. 88 FIG. 89 FIG. 90 FIG. 91

FIG. 92 FIG. 93 FIG. 94 FIG. 95

FIG. 96 FIG. 97 FIG. 98 FIG. 99

FIG. 100 FIG. 101 FIG. 102 FIG. 103

FIGS. 80–103. Line with auxiliary coördinate plane

DRILL SHEET NUMBER 5

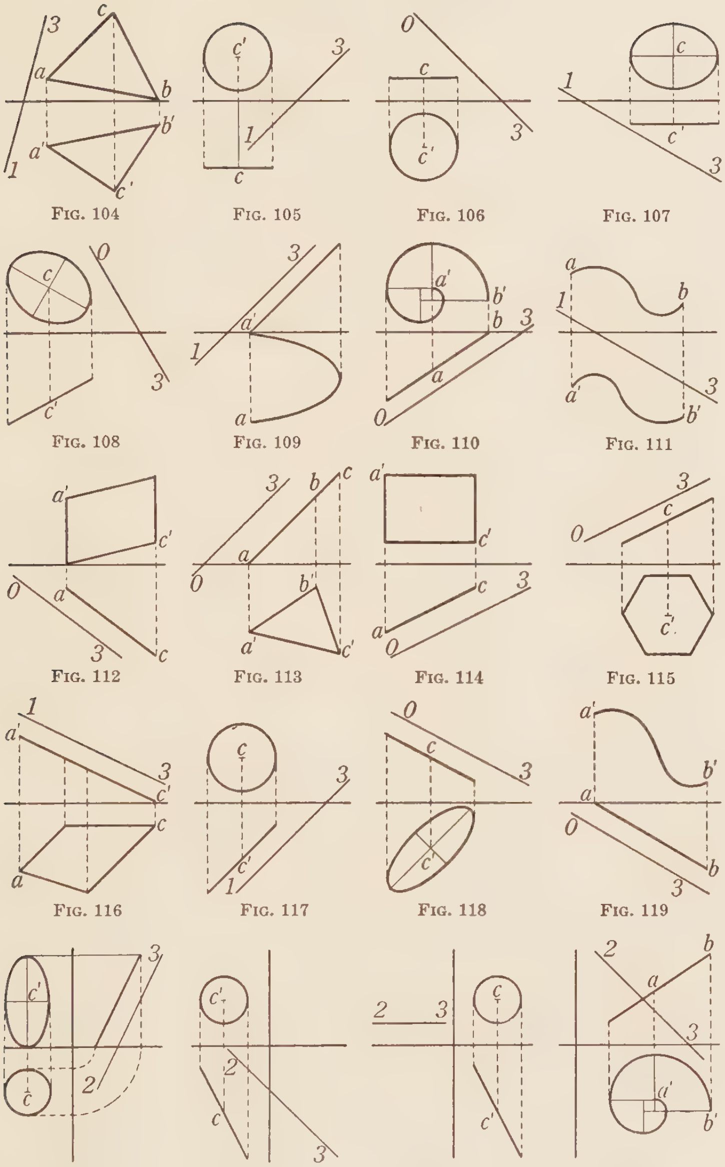


FIG. 104 FIG. 105 FIG. 106 FIG. 107

FIG. 108 FIG. 109 FIG. 110 FIG. 111

FIG. 112 FIG. 113 FIG. 114 FIG. 115

FIG. 116 FIG. 117 FIG. 118 FIG. 119

FIG. 120 FIG. 121 FIG. 122 FIG. 123

FIGS. 104–123. Plane figure with auxiliary coördinate plane

Drill Sheet Number 6

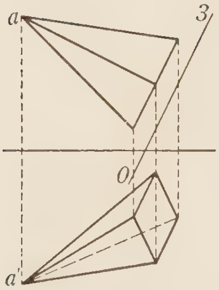


Fig. 124

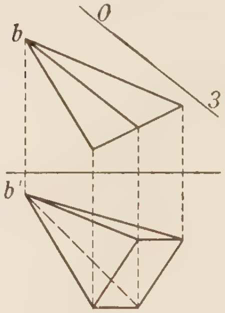


Fig. 125

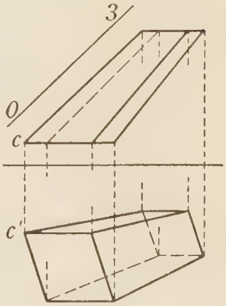


Fig. 126

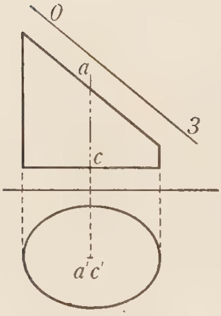


Fig. 127

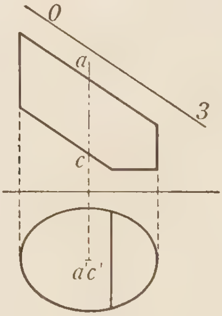


Fig. 128

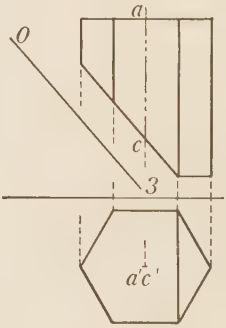


Fig. 129

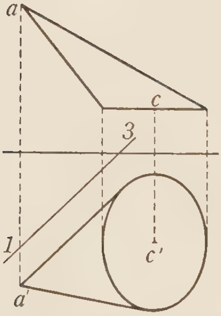


Fig. 130

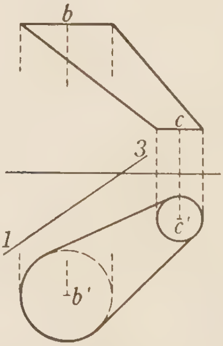


Fig. 131

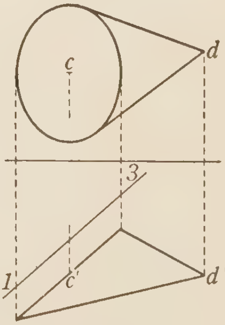


Fig. 132

Figs. 124–132. Solid object with auxiliary coördinate plane

DRILL SHEET NUMBER 7

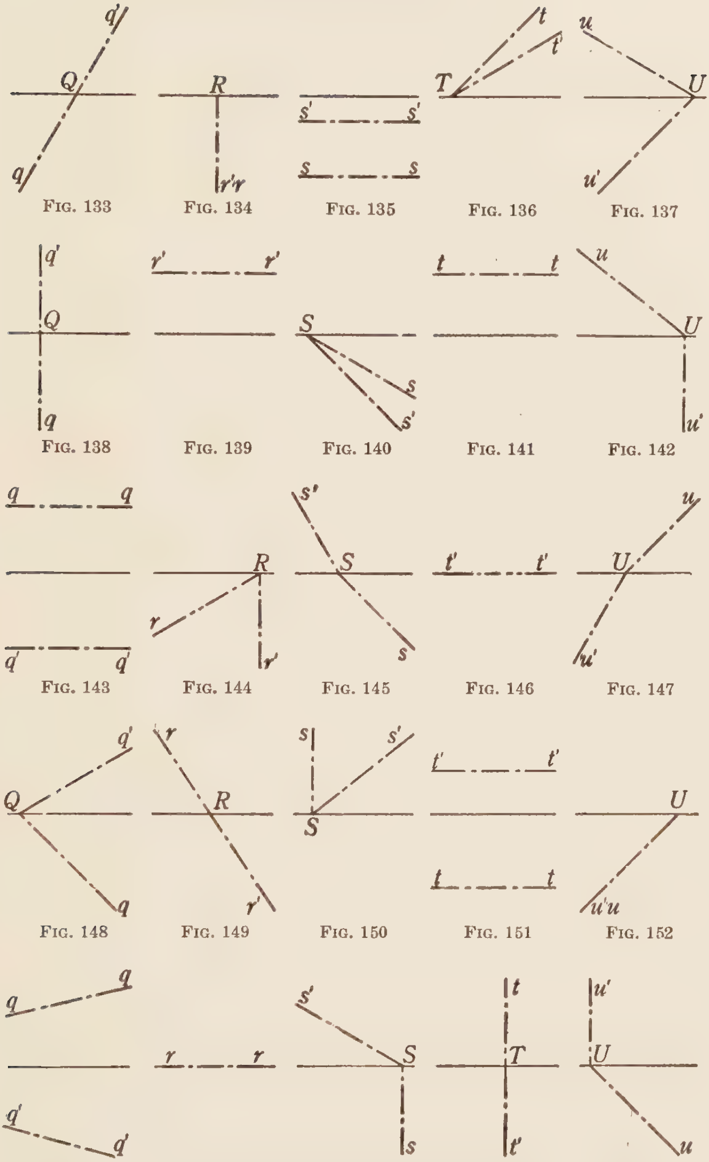


FIG. 133 FIG. 134 FIG. 135 FIG. 136 FIG. 137

FIG. 138 FIG. 139 FIG. 140 FIG. 141 FIG. 142

FIG. 143 FIG. 144 FIG. 145 FIG. 146 FIG. 147

FIG. 148 FIG. 149 FIG. 150 FIG. 151 FIG. 152

FIG. 153 FIG. 154 FIG. 155 FIG. 156 FIG. 157

FIGS. 133–157. Plane in space

DRILL SHEET NUMBER 8

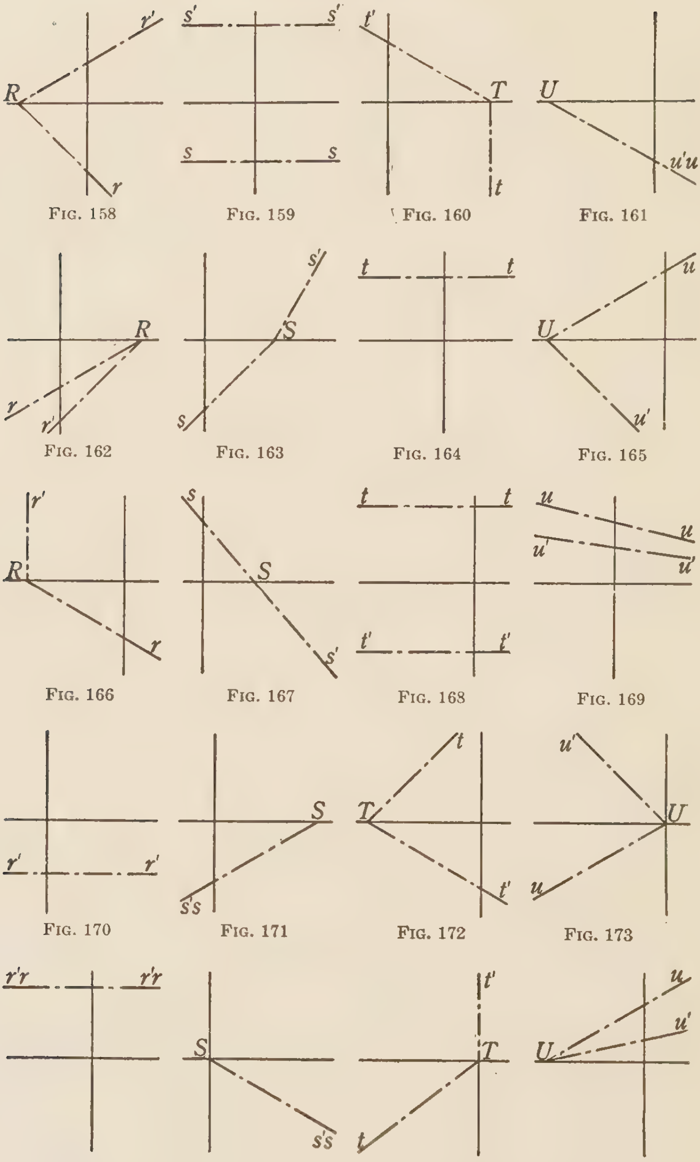


FIG. 158 FIG. 159 FIG. 160 FIG. 161

FIG. 162 FIG. 163 FIG. 164 FIG. 165

FIG. 166 FIG. 167 FIG. 168 FIG. 169

FIG. 170 FIG. 171 FIG. 172 FIG. 173

FIG. 174 FIG. 175 FIG. 176 FIG. 177

FIGS. 158–177. Plane with profile coördinate plane

Drill Sheet Number 9

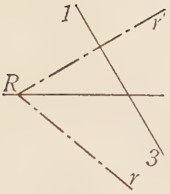


Fig. 178

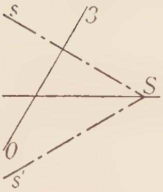


Fig. 179

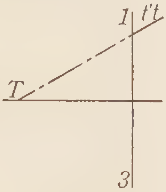


Fig. 180

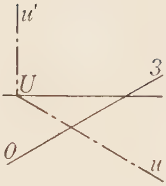


Fig. 181

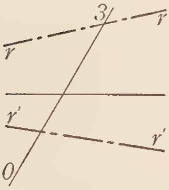


Fig. 182

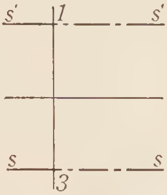


Fig. 183

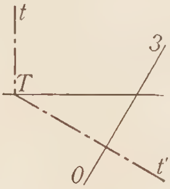


Fig. 184

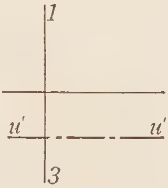


Fig. 185

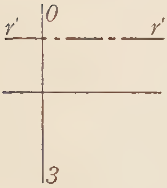


Fig. 186

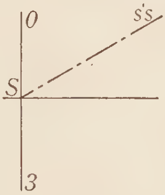


Fig. 187

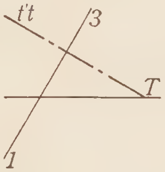


Fig. 188

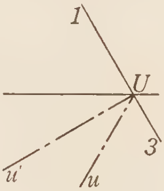


Fig. 189

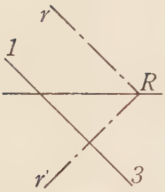


Fig. 190

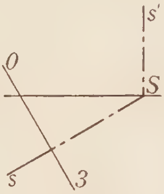


Fig. 191

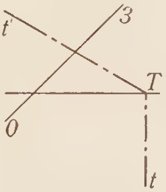
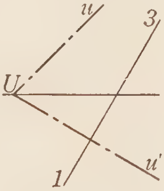


Fig. 192



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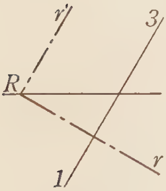

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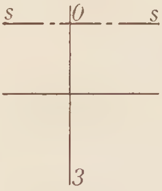


Fig. 195

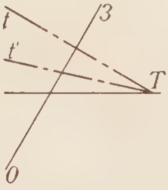


Fig. 196

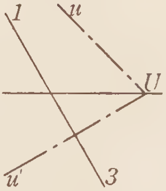


Fig. 197

Figs. 178–197. Plane with auxiliary coördinate plane

DRILL SHEET NUMBER 10

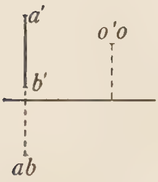


FIG. 198

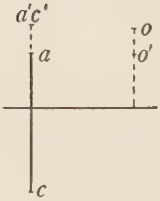


FIG. 199

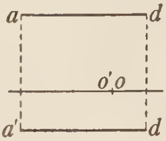


FIG. 200

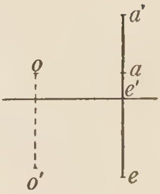


FIG. 201

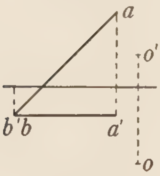


FIG. 202

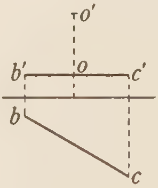


FIG. 203

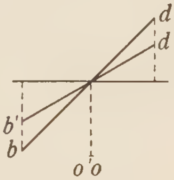


FIG. 204

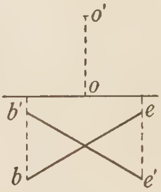


FIG. 205

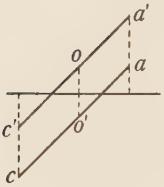


FIG. 206

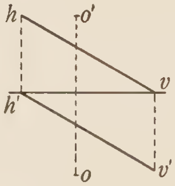


FIG. 207

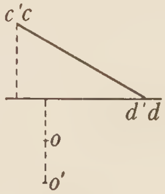


FIG. 208

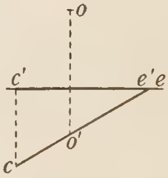


FIG. 209

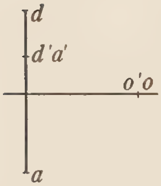


FIG. 210

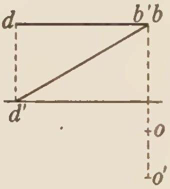


FIG. 211

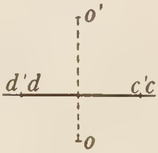


FIG. 212

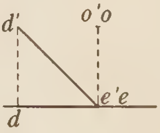


FIG. 213

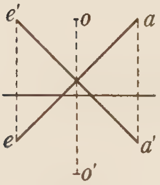


FIG. 214

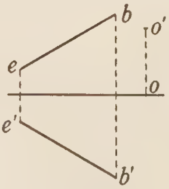


FIG. 215

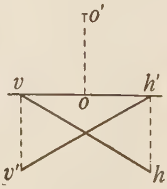


FIG. 216

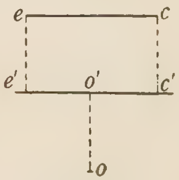


FIG. 217

FIGS. 198–217. Line and point

Drill Sheet Number 11

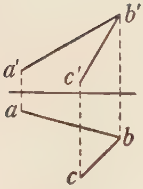


Fig. 218

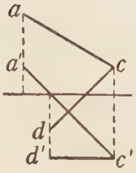


Fig. 219

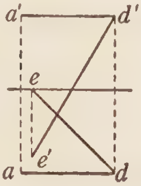


Fig. 220

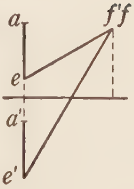


Fig. 221

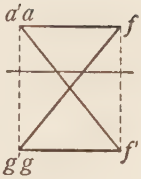
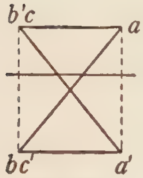


Fig. 222



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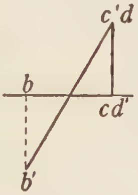


Fig. 224

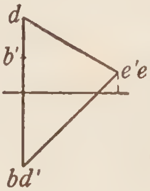


Fig. 225

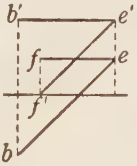


Fig. 226

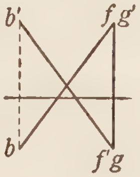


Fig. 227

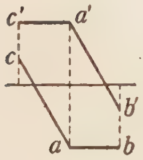
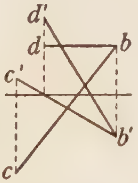


Fig. 228



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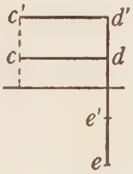


Fig. 230

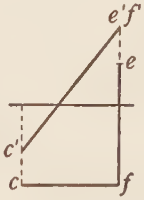


Fig. 231

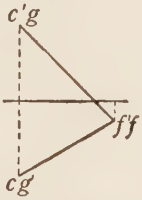


Fig. 232

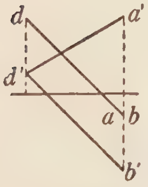


Fig. 233

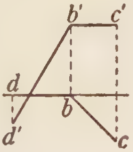


Fig. 234

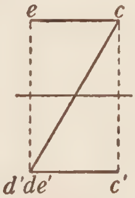


Fig. 235

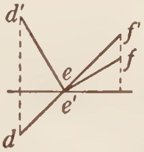


Fig. 236

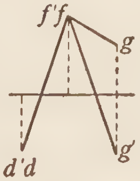


Fig. 237

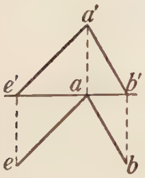


Fig. 238

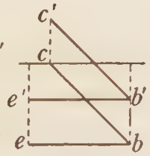


Fig. 239

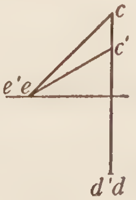


Fig. 240

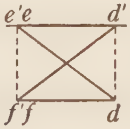


Fig. 241

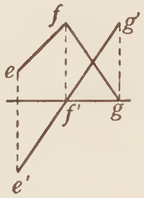


Fig. 242

Figs. 218–242. Two intersecting lines

DRILL SHEET NUMBER 12

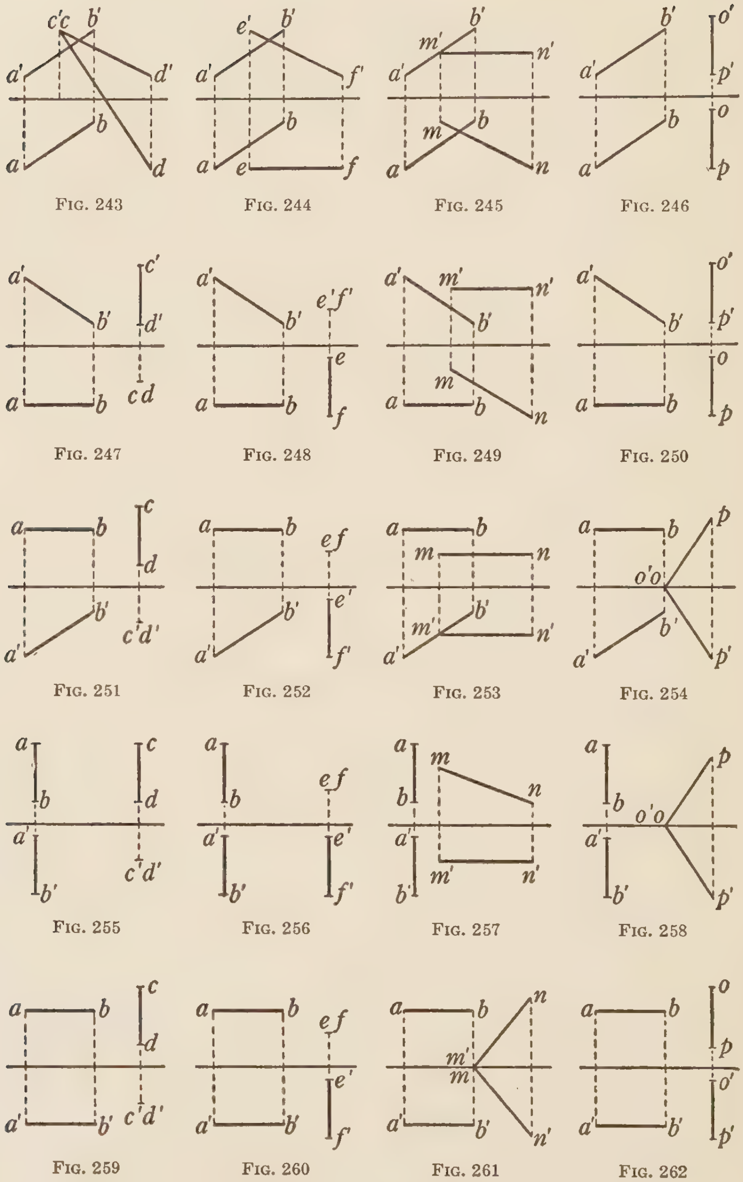


Fig. 243 Fig. 244 Fig. 245 Fig. 246

Fig. 247 Fig. 248 Fig. 249 Fig. 250

Fig. 251 Fig. 252 Fig. 253 Fig. 254

Fig. 255 Fig. 256 Fig. 257 Fig. 258

Fig. 259 Fig. 260 Fig. 261 Fig. 262

Figs. 243–262. Two non-intersecting, non-parallel lines

Drill Sheet Number 13

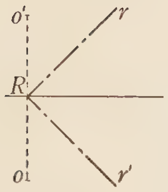


Fig. 263

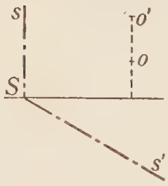


Fig. 264

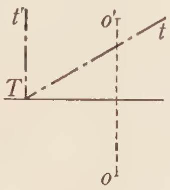


Fig. 265

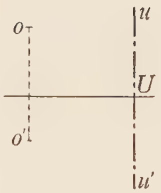


Fig. 266

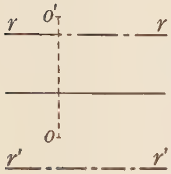


Fig. 267

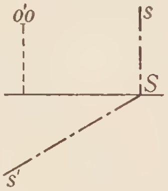


Fig. 268

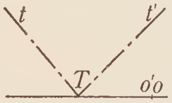


Fig. 269

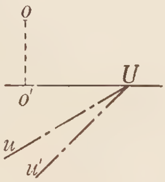


Fig. 270

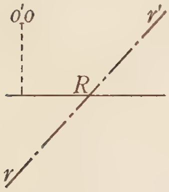


Fig. 271

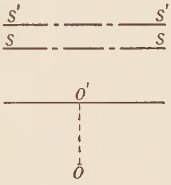


Fig. 272

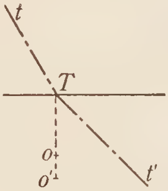


Fig. 273

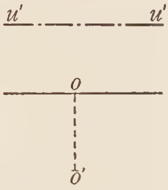


Fig. 274

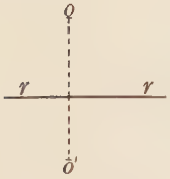


Fig. 275

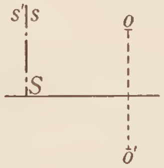


Fig. 276

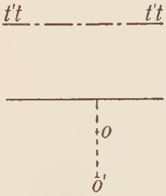


Fig. 277

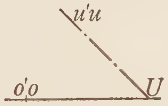


Fig. 278

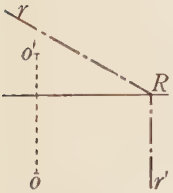


Fig. 279

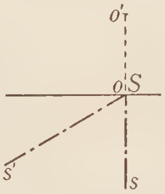


Fig. 280

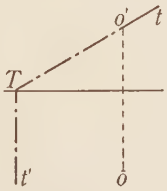


Fig. 281

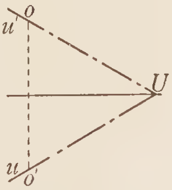


Fig. 282

Figs. 263–282. Plane and point

Drill Sheet Number 14

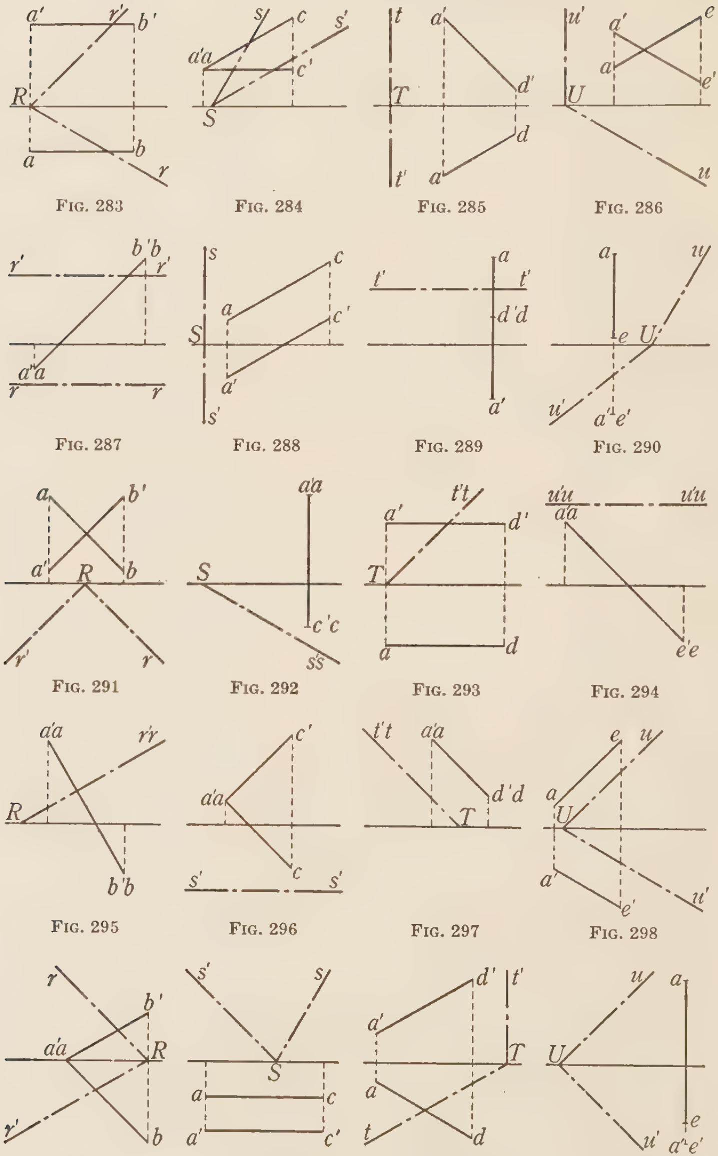


Fig. 283 Fig. 284 Fig. 285 Fig. 286

Fig. 287 Fig. 288 Fig. 289 Fig. 290

Fig. 291 Fig. 292 Fig. 293 Fig. 294

Fig. 295 Fig. 296 Fig. 297 Fig. 298

Fig. 299 Fig. 300 Fig. 301 Fig. 302

Figs. 283–302. Plane and line

Drill Sheet Number 15

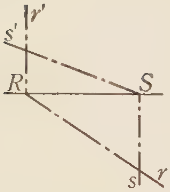


Fig. 303

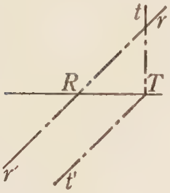


Fig. 304

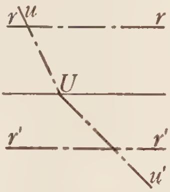


Fig. 305

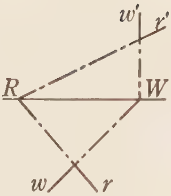


Fig. 306

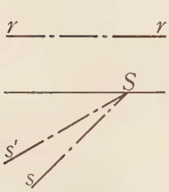
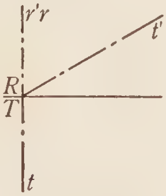


Fig. 307



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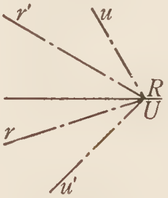


Fig. 309

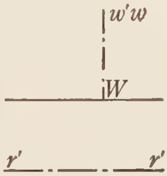


Fig. 310

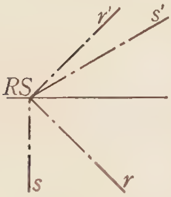


Fig. 311

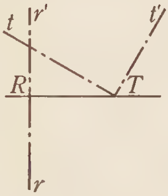


Fig. 312

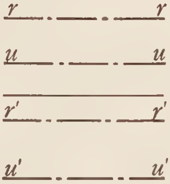


Fig. 313

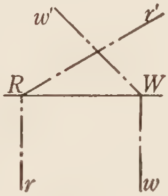


Fig. 314

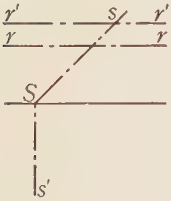


Fig. 315

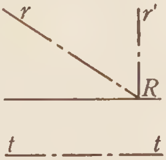


Fig. 316

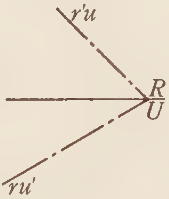


Fig. 317

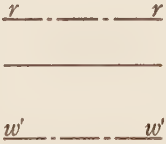


Fig. 318

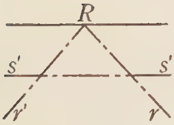


Fig. 319

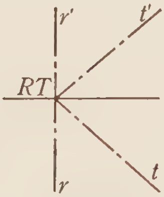


Fig. 320

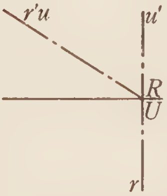


Fig. 321

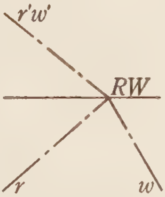


Fig. 322

Figs. 303–322. Two intersecting planes

Drill Sheet Number 16

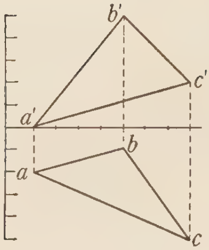


Fig. 323

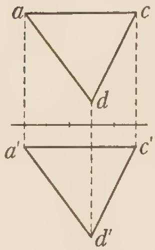


Fig. 324

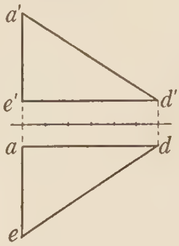


Fig. 325

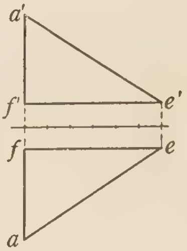


Fig. 326

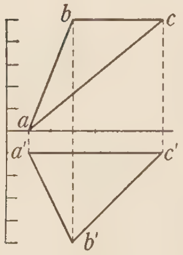


Fig. 327

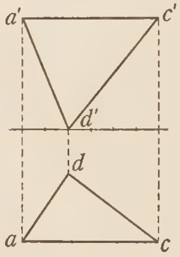


Fig. 328

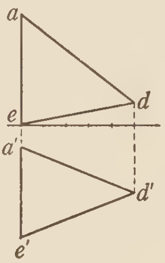


Fig. 329

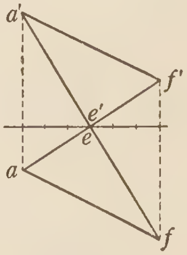


Fig. 330

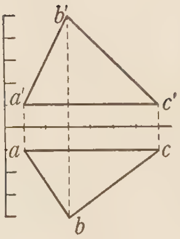


Fig. 331

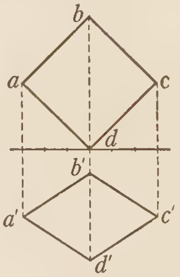


Fig. 332

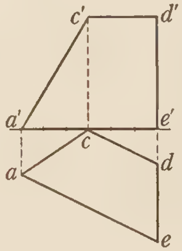


Fig. 333

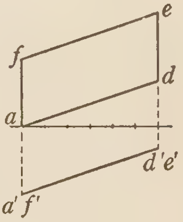


Fig. 334

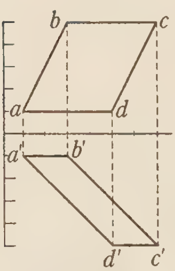


Fig. 335

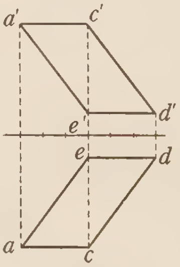


Fig. 336

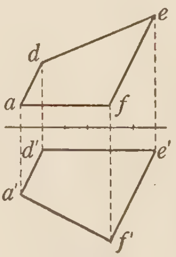


Fig. 337

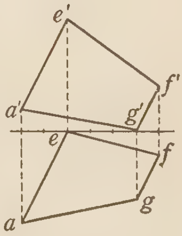
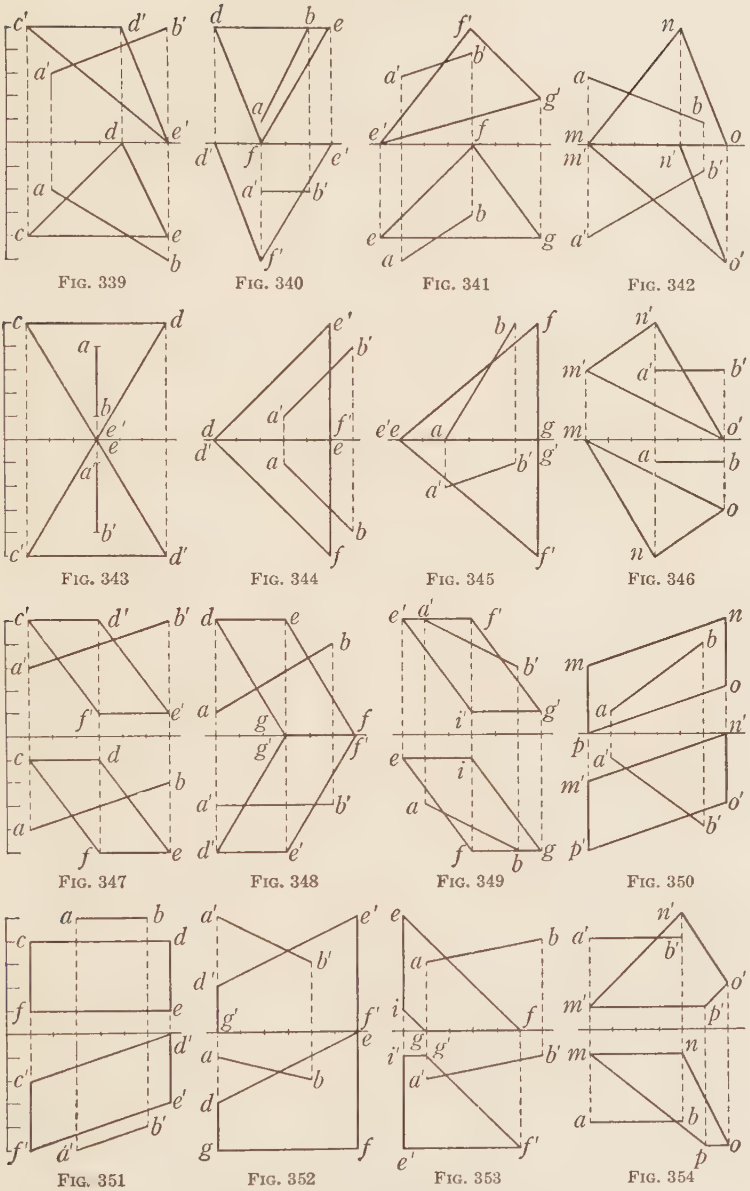


Fig. 338

Figs. 323–338. Plane triangles and quadrilaterals

DRILL SHEET NUMBER 17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d b e a d' f e' a' b' f'
f' b' a' g' e' f e b g a
n a b m o m' n' b' a' o'
c d a b e' e a' b' c' d'
e' b' a' d f' d' e a b f
b f e'e a g b' g' a' f'
n' m' a' b' m o' a b o n
c' d' b' a' f' e' c d b a f e
d e b a g f g' f' a' b' d' e'
e' a' f' b' i' g' e i a f b g
n m b o a p n' a' m' o' b' p'
a b c d f e d' c' e' b' f' a'
a' e' b' d' g' f' e a b d g f
e b a i f i' g g' b' a' f' e'
n' a' b' o' m' p' m n a b p o

FIG. 339 FIG. 340 FIG. 341 FIG. 342

FIG. 343 FIG. 344 FIG. 345 FIG. 346

FIG. 347 FIG. 348 FIG. 349 FIG. 350

FIG. 351 FIG. 352 FIG. 353 FIG. 354

FIGS. 339–354. Plane figure and line

DRILL SHEET NUMBER 18

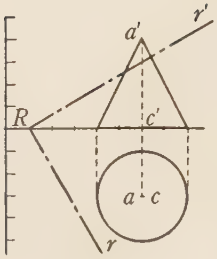

FIG. 355

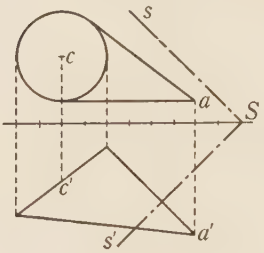

FIG. 356

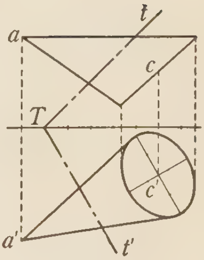

FIG. 357

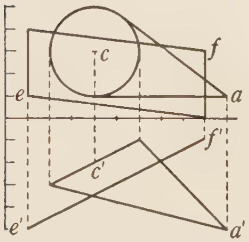

FIG. 358

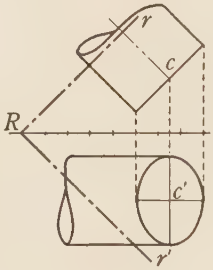

FIG. 359

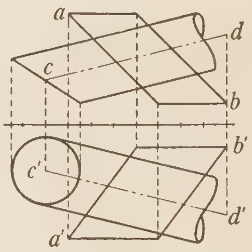

FIG. 360

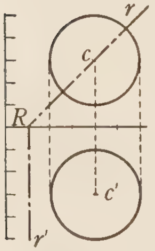

FIG. 361

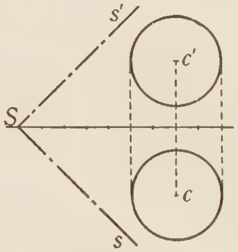

FIG. 362

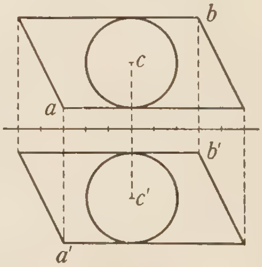

FIG. 363

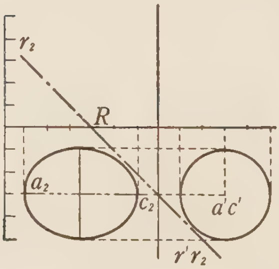

FIG. 364

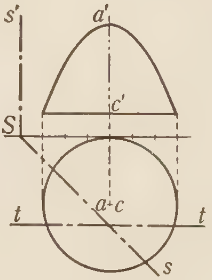

FIG. 365

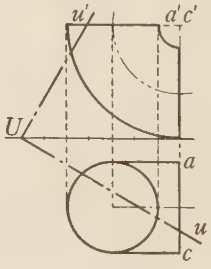

FIG. 366

FIGS. 355–366. Curved surface and intersecting plane

DRILL SHEET NUMBER 19

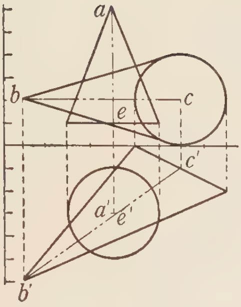

FIG. 367

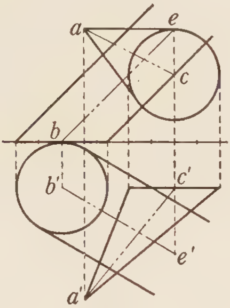

FIG. 368

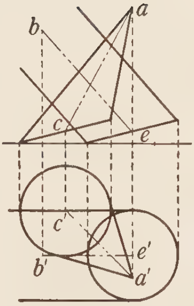

FIG. 369

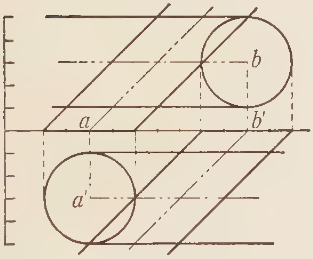

FIG. 370

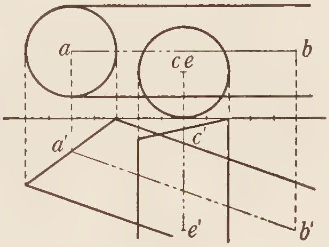

FIG. 371

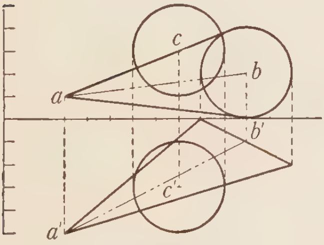

FIG. 372

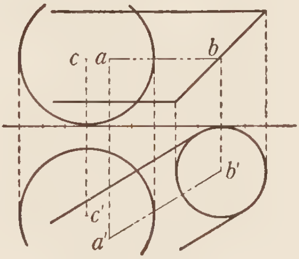

FIG. 373

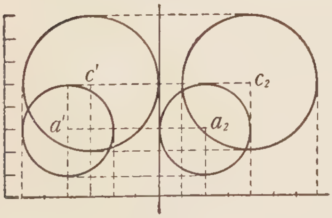

FIG. 374

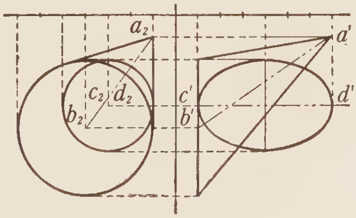

FIG. 375

FIGS. 367–375. Two intersecting curved surfaces

Articles 1 to 9. H, V, and P Projections of a Point

1. Plot a point in each of the four quadrants; in the V plane, both above and below the H plane; in the H plane, both behind and in front of the V plane; and in both the H and V planes. What are these nine points called when taken as a group?

2. Find the profile projection of any assigned point shown in Figs. 1 to 15, page 265.

3. Plot the H, V, and P projections of any assigned point from the following: A$(-2, \frac{1}{4}, 1\frac{1}{4})$; B$(-1\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$; C$(-1, \frac{3}{4}, -1)$; D$(-\frac{1}{2}, -1, -\frac{1}{2})$; E$(0, 1\frac{1}{8}, 0)$; F$(\frac{1}{2}, 0, 1\frac{1}{4})$; G$(1, 0, 0)$; H$(1\frac{1}{2}, 0, -1)$.

4. Plot the point M (1, 1, 1). Locate the point N $\frac{1}{2}''$ nearer the P plane, $\frac{1}{4}''$ nearer the H plane, and $\frac{1}{4}''$ nearer the V plane than the point M, and in the same quadrant.

5. Plot the H, V, and P projections of the point A$(\frac{1}{2}, 1\frac{1}{4}, 1)$, and of the point B, which is $1''$ to the left of, $\frac{1}{2}''$ lower than, and $\frac{1}{2}''$ to the rear of the point A.

6. Plot the H, V, and P projections of the point A$(\frac{3}{4}, \frac{1}{2}, \frac{7}{8})$, and of the point B, which is on the opposite side of the H, V, and P planes and as far from each as the point A.

7. Complete the three principal projections of the points of Fig. 376, using the following additional data: for the point A, $z = \frac{1}{2}''$; for the point B, $x = -\frac{1}{2}''$; for the point C, $y = 1.''$

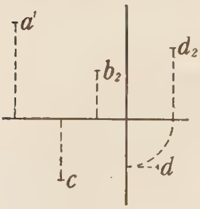

FIG. 376

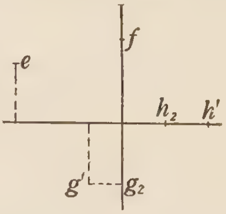

FIG. 377

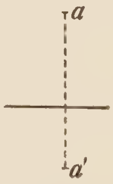

FIG. 378

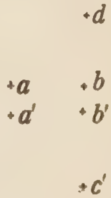

FIG. 379

8. Complete the three principal projections of the points of Fig. 377, using the following additional data: the point E is in the H plane; for the point F, $y = \frac{1}{2}''$.

9. Plot the points B, C, and D at the same elevation as the point A, Fig. 378, and on the three corners of an imaginary one-inch square, of which point A is the fourth corner and two sides of which are parallel to the V plane.

10. Plot the points E, F, G, and K at the corners of an imaginary one-inch cube, of which points A, B, C, and D, Fig. 379, are the other four corners.

Article 10. The Auxiliary-Coördinate-Plane-3 Projection of a Point

11. Find the auxiliary-coördinate-plane-3 projection of any assigned point shown in Figs. 16 to 30, page 265.

Articles 11 to 13. The H, V, and P Projections of Lines

12. Find the profile projection of any assigned line shown in Figs. 31 to 54, page 266.

13. Draw a straight line ABC with the points A, B, and C in the first, second, and third quadrants respectively.

14. Draw the line AB from A$(-\frac{1}{2}, 1\frac{1}{4}, \frac{1}{8})$ to B$(\frac{3}{4}, -1, \frac{3}{4})$, and on it locate the point C, $\frac{1}{2}''$ above the H plane, and the point D, $\frac{1}{2}''$ in front of the V plane.

15. The line AB, 1″ long, is the base of an isosceles triangle ABC, Fig. 380. The triangle is parallel to the H plane. Draw the two principal views of the triangle.

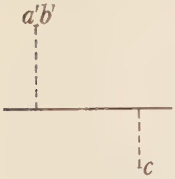

FIG. 380

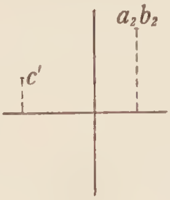

FIG. 381

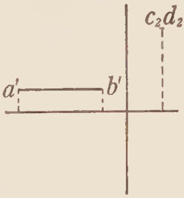

FIG. 382

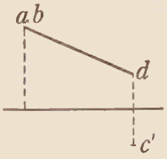

FIG. 383

16. The line AB, 1″ long, is one of the equal sides of an isosceles triangle ABC, Fig. 381. The triangle is parallel to the V plane. Draw the three principal views of the triangle.

17. Complete the three views of the rectangle ABCD, Fig. 382, that is parallel to the V plane.

18. Complete the H and V projections of the square ABCD, Fig. 383.

19. Complete the two views of the square ABCD, Fig. 384.

20. Complete the three views of the square ABCD, Fig. 385.

21. Complete the three views of the square ABCD, Fig. 386.

22. Complete the three views of the square ABCD, Fig. 387.

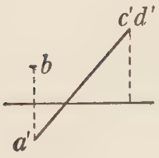

FIG. 384

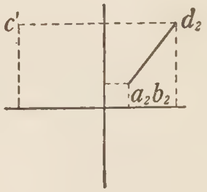

FIG. 385

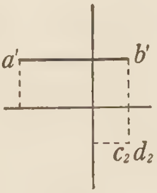

FIG. 386

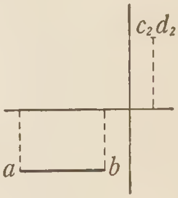

FIG. 387

Articles 14 and 15. The Relations of Lines to Coördinate Planes; Points on Lines

23. Through any assigned point shown in Figs. 1 to 15, page 265, draw

(*a*) a line inclined to the three principal coördinate planes;

(*b*) a line parallel to the H plane and inclined to the V and P planes;

(*c*) a line parallel to the V plane and inclined to the H and P planes;

(*d*) a line parallel to the P plane and inclined to the H and V planes;

(*e*) three lines perpendicular, respectively, to the three principal coördinate planes.

24. Draw each of the following lines: (*a*) AB, in the first quadrant, parallel to the H plane and inclined 30° to the V plane; (*b*) CD, in the third quadrant, parallel to the V plane and inclined 45° to the H plane; (*c*) EF, in the second quadrant, parallel to the P plane and inclined 30° to the H plane.

25. If any assigned line shown in Figs. 55 to 79, page 267, were extended indefinitely in both directions, state which quadrants would contain it.

26. Locate two points, O and P, in the line AB, Fig. 388, that are equidistant from the H and V planes. In which quadrants are they?

27. Locate two points, O and P, in the line CD, Fig. 389, that are equidistant from the V and P planes. In which quadrants are they?

28. Locate two points, O and P, in the line EF, Fig. 390, that are equidistant from the H and P planes. In which quadrants are they?

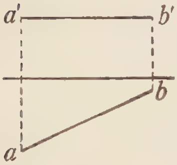

FIG. 388

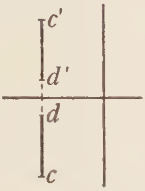

FIG. 389

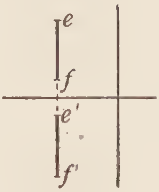

FIG. 390

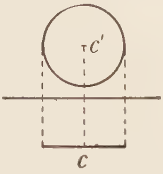

FIG. 391

29. On the circumference of the circle whose center is the point C, Fig. 391, locate two points, O and P, that are equidistant from the H and V planes.

Article 16. Intersecting Lines

30. Select an intermediate point O on any assigned line shown in Figs. 55 to 79, page 267, and draw a line OM of any length and inclined to the three principal coördinate planes.

31. Prove by construction and lettering that the following lines of Fig. 392 do or do not intersect: (*a*) AB and CD; (*b*) AB and EF; (*c*) CD and EF when produced.

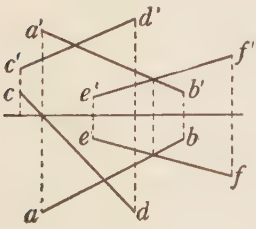

FIG. 392

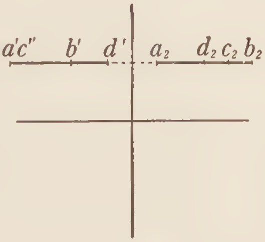

FIG. 393

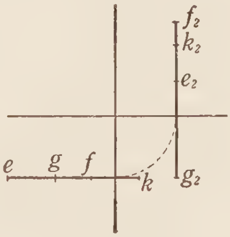

FIG. 394

32. Plot the point $O(x, -1\frac{1}{4}, -1\frac{1}{2})$. From O draw a line OM parallel to the H plane and inclined 45° to the V plane; draw a line ON parallel to the V plane and inclined 30° to the H plane. Join the points M and N to complete the triangle MNO.

33. Find the point M in which the lines AB and CD, Fig. 393, intersect. In which quadrant is M?

34. Find the point M in which the lines EF and GK, Fig. 394, intersect. In which quadrant is M?

35. Do the lines AB and CD, Fig. 395, intersect? If so, locate their point of intersection M.

36. The line CD, parallel to the H plane, intersects the line AB at the point E, Fig. 396. Draw the H and V projections of the two lines.

37. The lines AB and CD intersect in the point E, Fig. 397. Draw the H, V, and P projections of the two lines.

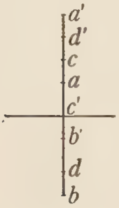

FIG. 395

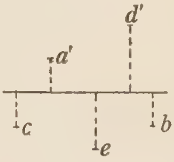

FIG. 396

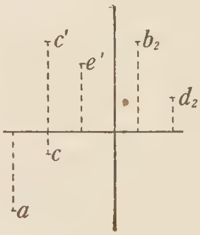

FIG. 397

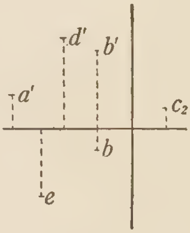

FIG. 398

38. The lines AB and CD intersect in the point E, Fig. 398. Draw the H, V, and P projections of the two lines.

39. Construct the triangle MNO by joining the midpoints of the lines AB, CD, and EF, Fig. 399, with straight lines.

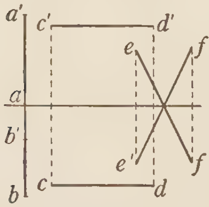

FIG. 399

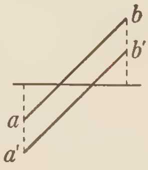

FIG. 400

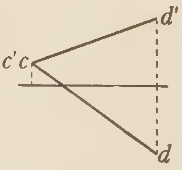

FIG. 401

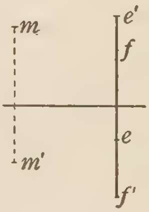

FIG. 402

40. Locate a point M in the line AB, Fig. 400, $\frac{1}{2}''$ behind the V plane, and a point N $\frac{3}{4}''$ below the H plane. Draw a line MO parallel to the V plane. Connect points N and O.

41. Locate a point O in the line CD, Fig. 401, $\frac{1}{2}''$ above the H plane. How far is it from the V plane? Draw a line ON parallel to the H plane.

42. Draw a line MN parallel to the H plane and intersecting the line EF in a point N, Fig. 402.

Article 17. The Auxiliary-Coördinate-Plane-3 Projection of a Line

43. Find the auxiliary-coördinate-plane-3 projection of any assigned line shown in Figs. 80 to 103, page 268.

44. Find the auxiliary-coördinate-plane-3 projection of any assigned plane geometrical figure shown in Figs. 104 to 123, page 269.

45. Find the H and the auxiliary-coördinate-plane-3 projections of the line AB, Fig. 403.

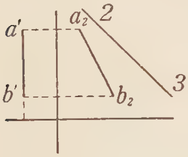

FIG. 403

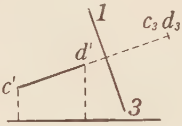

FIG. 404

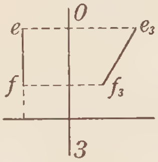

FIG. 405

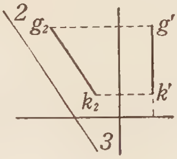

FIG. 406

46. The line CD, Fig. 404, is in the first quadrant. Find its H projection.

47. The line EF, Fig. 405, is in the third quadrant. Find its V projection.

48. Find the H and the auxiliary-coördinate-plane-3 projections of the line GK, Fig. 406.

Article 18. Points in which Lines Pierce the Coördinate Planes

49. Find the H and V piercing points of any assigned line shown in Figs. 55 to 79, page 267. If the line is parallel to a coördinate plane, thus having its piercing point in that plane at infinity, a neatly printed statement of this fact should be included in the solution.

50. Find the P piercing point of any assigned line shown in Figs. 31 to 54, page 266. If the line is parallel to the P plane, thus having its piercing point in that plane at infinity, a neatly printed statement of this fact should be given as the solution.

51. The line AB, Fig. 407, pierces the H, V, and P planes at the same point. Draw the three projections of AB and letter its three piercing points.

52. The line CD, Fig. 408, pierces the H and V planes at the same point. Draw the H and V projections of CD and letter its two piercing points.

53. Draw the straight line ABC, which pierces the H plane at B, Fig. 409, and letter its H and V piercing points.

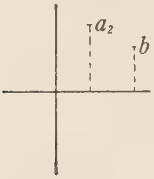

FIG. 407

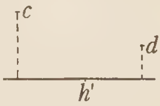

FIG. 408

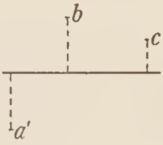

FIG. 409

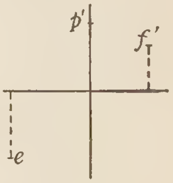

FIG. 410

54. The straight line EF, Fig. 410, pierces the V and P planes at the same point P. Draw its H, V, and P projections and letter its three piercing points.

Article 19. The True Length of a Line

55. Find the true length of any assigned line shown in Figs. 55, 57, 60, 61, 62, 63, 67, 68, 69, 71, 76, 78, 79, page 267, by using an auxiliary coördinate plane parallel to the line.

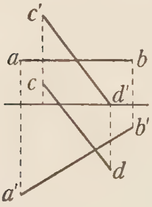

FIG. 411

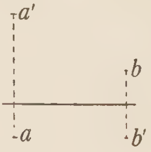

FIG. 412

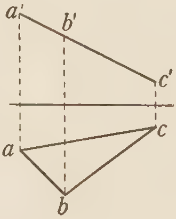

FIG. 413

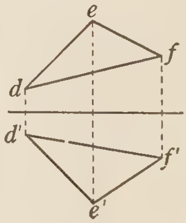

FIG. 414

56. Find the true size of any assigned plane geometrical surface shown in Figs. 112 to 120, page 269, by projecting it upon the auxiliary coördinate plane indicated, which is parallel to the given surface.

57. Which line of Fig. 411 is the longer, and how much?

58. Find the distance from the point A to the point B, Fig. 412.

59. Show by constructions and dimensions the true lengths of the sides of the triangle ABC, Fig. 413.

60. Find the true length of each side of the triangle DEF, Fig. 414, by means of auxiliary views. Construct the triangle from these true lengths.

61. The points B and C, Fig. 415, are equidistant from the point A. Find the V projection of the point B.

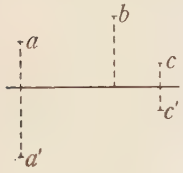

FIG. 415

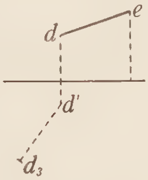

FIG. 416

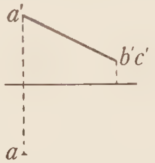

FIG. 417

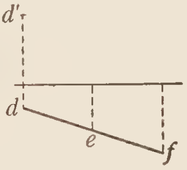

FIG. 418

62. The line DE, Fig. 416, projects in true length upon a coördinate plane 3 perpendicular to the V plane. Determine the ground line 1–3, the V and plane-3 projections of DE, and the H and V piercing points of DE.

63. Draw the H projection of the equilateral triangle ABC, Fig. 417.

64. Draw the V projection of the equilateral triangle DEF, Fig. 418.

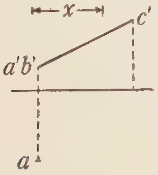

FIG. 419

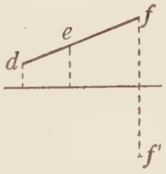

FIG. 420

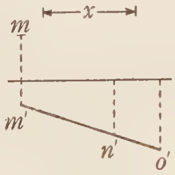

FIG. 421

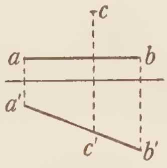

FIG. 422

65. An isosceles triangle ABC, Fig. 419, has an altitude equal to the length x shown, with BC as the base. Construct the H projection of the triangle.

66. An isosceles triangle DEF, Fig. 420, has an altitude equal to *df*, with EF as the base. Construct the V projection of the triangle.

67. The line NO, Fig. 421, equal to the distance *x* in length, is the base of an isosceles triangle MNO. Construct the H projection of the triangle.

68. The line AB, Fig. 422, is tangent to the circle whose center is the point C. Construct 'the H and V projections of the circle.

Articles 20 and 22. The Angle between Lines; Perpendicular Lines

69. Select a point O on any assigned line shown in Figs. 56, 58, 59, 62, 64, 65, 66, 71, 72, 73, 74, 75, 77, and 78, page 267, and draw lines OM, ON, and OP, making angles of (*a*) 30°, (*b*) 45°, (*c*) 60°, respectively, with the given line.

70. Select a point O on any assigned line shown in Figs. 55 to 79, page 267, and draw a line OM making an angle of 90° with the given line.

Article 21. Parallel Lines

71. Draw a line OM through the given point O and parallel to the given line in any assigned set shown in Figs. 198 to 217, page 274.

72. Draw the line CD parallel to the line AB, Fig. 423.

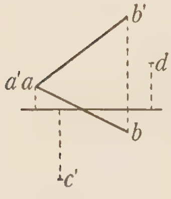

FIG. 423

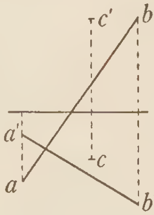

FIG. 424

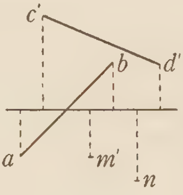

FIG. 425

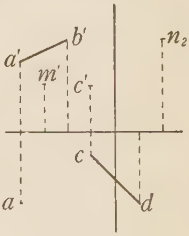

FIG. 426

73. The point C, Fig. 424, is the midpoint of a line MN, which is parallel to AB and of the same length. Draw the line MN.

74. The line AB through the point M, Fig. 425, is parallel to the line CD through the point N. Complete the projections of the two points and lines.

75. The lines AB, CD, and MN, Fig. 426, are parallel. Find the three principal views of the three lines.

Article 23. Representation of Solid Objects; Determination of Quadrants

76. State the quadrant in which any assigned object shown in Figs. 427 to 436 is represented, and redraw the object in the diagonally opposite quadrant, showing three views.

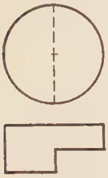
FIG. 427

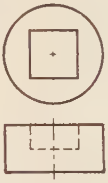
FIG. 428

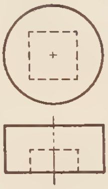
FIG. 429

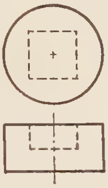
FIG. 430

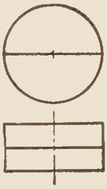
FIG. 431

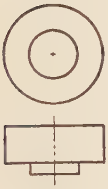
FIG. 432

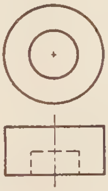
FIG. 433

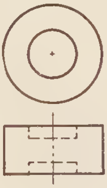
FIG. 434

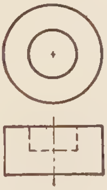
FIG. 435

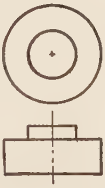
FIG. 436

77. Draw the profile view of the triangular prism shown in Fig. 437, which is in the third quadrant.

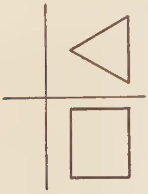
FIG. 437

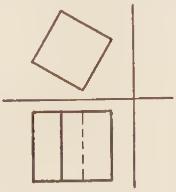
FIG. 438

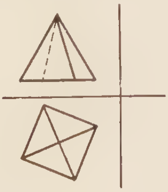
FIG. 439

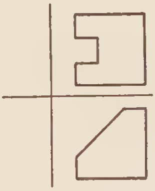
FIG. 440

78. Draw the profile view of the square prism shown in Fig. 438, which is in the first quadrant.

79. Draw the profile view of the square pyramid shown in Fig. 439, which is in the first quadrant.

80. Complete the top and front views of the object shown in Fig. 440, and draw a left side view. The object is in the third quadrant.

81. Draw the top view of the tetrahedron shown in Fig. 441. Letter all corners.

82. The hexagonal prism shown in Fig. 442 is in the first quadrant, its axis being perpendicular to the V plane. Draw the H projection of the prism. Letter all its corners.

83. The object shown in Fig. 443 is represented in the third quadrant. Draw a left side view of the object.

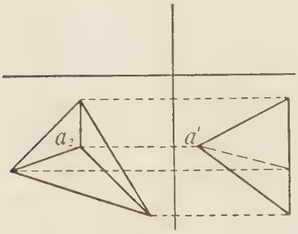

FIG. 441

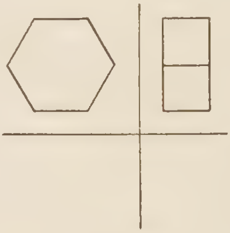

FIG. 442

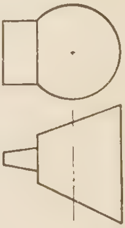

FIG. 443

84. Find the auxiliary-coördinate-plane-3 projection of any assigned object shown in Figs. 124 to 132, page 270.

85. Draw three views of a right triangular prism with faces 1″ × 2″, $\frac{1}{2}$″ × 2″, and $\frac{3}{4}$″ × 2″, the prism to be placed in the third quadrant with its axis perpendicular to the V plane, and the 1″ × 2″ face parallel to the H plane.

86. Draw three views of the prism of Problem 85 placed in the third quadrant with its lateral edges perpendicular to the H plane and the 1″ × 2″ face inclined 30° to the V plane.

87. A right circular cylinder, 2″ in diameter and 3″ long, is placed in the third quadrant with its axis perpendicular to the H plane. Draw the H, V, and P projections of the cylinder.

88. Draw three views of the cylinder of Problem 87 placed in the third quadrant with its axis perpendicular to the V plane.

89. Draw three views of the cylinder of Problem 87 placed in the third quadrant with its axis perpendicular to the P plane.

90. Join each point in Fig. 444 to the other three points by straight lines. The object thus formed is a tetrahedron. Show the hidden edges as broken lines.

91. Complete the V projection of the right pyramid, Fig. 445, of which the square ABCD is the base.

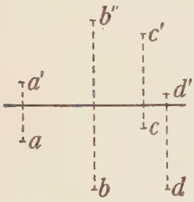

FIG. 444

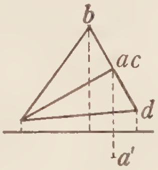

FIG. 445

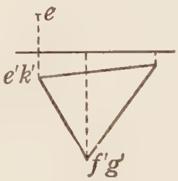

FIG. 446

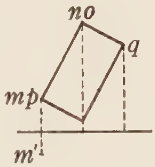

FIG. 447

92. Complete the H projection of the right pyramid, Fig. 446, of which the square EFGK is the base.

93. Complete the V projection of the right prism, Fig. 447, of which the square MNOP is one base and the point Q is one corner of the other base.

94. The object shown in Fig. 448 is represented in the third quadrant. Draw a left side view of the object.

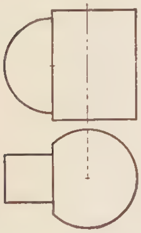

FIG. 448

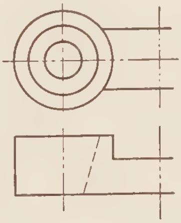

FIG. 449

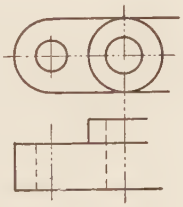

FIG. 450

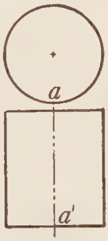

FIG. 451

95. The partial views shown in Fig. 449 are of an object which has two planes of symmetry. The object is represented in the third quadrant. Complete both views and draw a left side view.

96. The partial views shown in Fig. 450 are of an object which has two planes of symmetry. The object is represented in the third quadrant. Complete both views and draw a left side view.

97. The point A moves around the cylinder, Fig. 451, and upward at uniform velocities, reaching the top in two revolutions. Plot both projections of A at each 30-degree interval of rotation and connect each set of projections by a smooth curve. The space curve traversed by A is a cylindrical helix. Its H projection is a circle and its V projection is a sine curve.

Articles 24 to 28. Planes and their Trace Representations

98. Draw the H, V, and P traces of seven characteristic planes in the third quadrant, constituting the alphabet of a plane in that quadrant.

99. Find the profile trace of any assigned plane shown in Figs. 158 to 177, page 272.

100. Draw the H and V traces of the planes described below, each plane being limited to the third quadrant:

(*a*) Plane Q, perpendicular to the H plane and inclined 30° to the V plane.

(*b*) Plane R, perpendicular to the V plane and inclined 60° to the H plane.

(*c*) Plane S, perpendicular to the P plane and inclined 45° to the H and V planes.

(*d*) Plane T, parallel to the P plane.

(*e*) Plane U, parallel to the V plane.

(*f*) Plane W, parallel to the H plane.

(*g*) Plane X, inclined to the H, V, and P planes.

101. Draw the H and V traces of the plane R, Fig. 452. Extend the plane into four quadrants and show the proper lettering of traces.

102. Draw the V trace of the plane S, Fig. 453. Extend the plane into four quadrants and show the proper lettering of traces.

103. Draw the V trace of the plane T, Fig. 454. Extend the plane into four quadrants and show the proper lettering of traces.

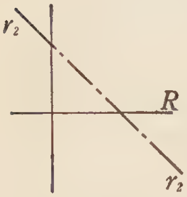

FIG. 452

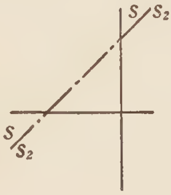

FIG. 453

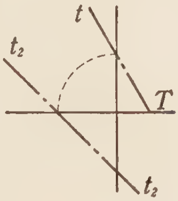

FIG. 454

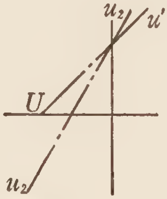

FIG. 455

104. Draw the H trace of the plane U, Fig. 455. Extend the plane into four quadrants and show the proper lettering of traces.

Article 29. Parallel Planes

105. Pass a plane M parallel to any assigned plane shown in Figs. 158 to 177, page 272, and find the profile trace of the plane so passed.

Article 30. The Auxiliary-Coördinate-Plane-3 Trace of a Plane

106. Find the trace of any assigned plane shown in Figs. 178 to 197, page 273, on the auxiliary coördinate plane indicated.

Article 33. A Point or Line in an Inclined Plane

107. Draw a line AB (VH where possible) inclined to at least two coördinate planes and in any assigned plane shown in Figs. 133 to 157, page 271.

108. Assume a point O near the center and in the plane of any assigned triangle or quadrilateral shown in Figs. 323 to 338, page 280. (See Article 31.)

109. Find the missing H or V projection of the following lines: (*a*) the line VH lying in the plane S, Fig. 456; (*b*) the line AB lying in the plane T, Fig. 457; (*c*) the line CD lying in the plane U, Fig. 458; (*d*) the line EF lying in the plane W, Fig. 459.

110. The plane R of Fig. 460 contains the points A and B. Determine the V projection of A and the H projection of B.

111. The line AB, Fig. 461, contains the point C and lies in the plane S. Determine the V projection of AB, the H projection of C, and the H trace of S.

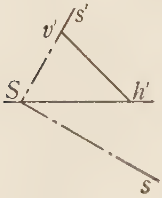

FIG. 456

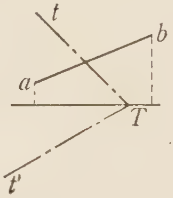

FIG. 457

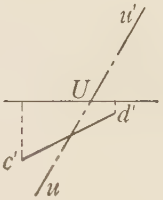

FIG. 458

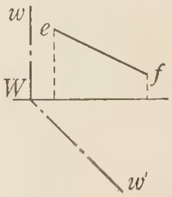

FIG. 459

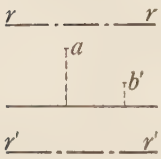

FIG. 460

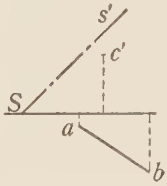

FIG. 461

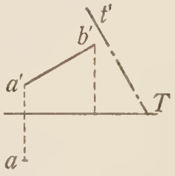

FIG. 462

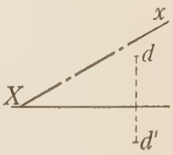

FIG. 463

112. The line AB, Fig. 462, lies in the plane T. Find the H projection of AB and the H trace of T.

113. The plane X, Fig. 463, contains the point D. Determine the V trace of X.

114. The lines EF and MN lie in the plane of the lines AB and BC, Fig. 464. Find their H projections without locating the traces of the plane.

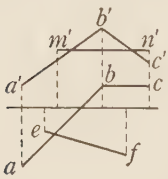

FIG. 464

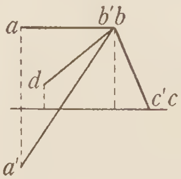

FIG. 465

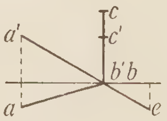

FIG. 466

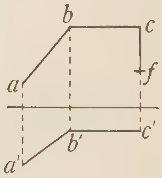

FIG. 467

115. The line BD lies in the plane of the lines AB and BC, Fig. 465. Find its V projection without locating the traces of the plane.

116. The line BE lies in the plane of the lines AB and BC, Fig. 466. Find its V projection without locating the traces of the plane.

117. The line CF lies in the plane of the lines AB and BC, Fig. 467. Find its V projection without locating the traces of the plane.

118. The point O lies in the plane of the trapezoid ABCD, Fig. 468. Complete the H projection of ABCD.

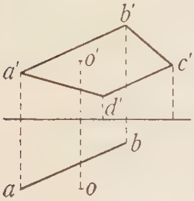

Fig. 468

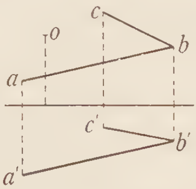

Fig. 469

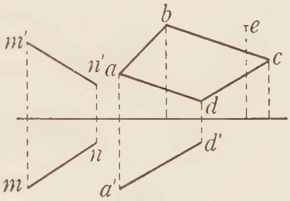

Fig. 470

119. Draw the line BO in the plane of the lines AB and BC, Fig. 469, without finding the traces of the plane.

120. A line EF in the plane of the trapezoid ABCD, Fig. 470, is parallel to the line MN. Draw the line EF and the V projection of ABCD.

Article 34. H-, V-, and P- Parallels of a Plane

121. Draw a line parallel to the H plane in any assigned plane shown in Figs. 133 to 157, page 271.

122. Draw a line parallel to the V plane in any assigned plane shown in Figs. 133 to 157, page 271.

123. Draw a line parallel to the P plane in any assigned plane shown in Figs. 133 to 157, page 271.

124. Draw a line parallel to the H plane and a line parallel to the V plane, both lying in the plane of the figure and through a point O near the midpoint of any assigned polygon shown in Figs. 323 to 338, page 280.

125. Draw the H, V, and P projections of a triangle ABC, of which the side AB is an H-parallel, the side BC is a V-parallel, and the side CA is a P-parallel.

126. Draw the line AB from A($0, 1, 1\frac{1}{8}$) to B($1\frac{1}{8}, \frac{1}{4}, \frac{1}{2}$) and the point C($\frac{1}{4}, \frac{3}{8}, \frac{1}{4}$). Draw (*a*) an H-parallel CD intersecting AB at the point D; (*b*) a V-parallel CE intersecting AB at the point E; and (*c*) a P-parallel CF intersecting AB at the point F.

127. Without finding the traces, but in the plane of the lines AB and AD, Fig. 471, draw (*a*) an H-parallel AM; (*b*) a V-parallel CN; (*c*) a P-parallel BO.

128. Without finding the traces, but in the plane of the lines AB and BC, Fig. 472, draw (*a*) AF parallel to the H plane; (*b*) CG parallel to the V plane; and (*c*) CK parallel to the P plane.

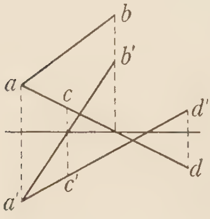

FIG. 471

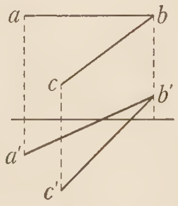

FIG. 472

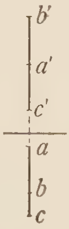

FIG. 473

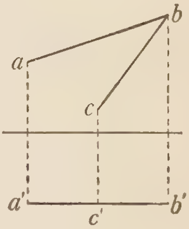

FIG. 474

129. Find the true length of the H-parallel AE contained in the triangle ABC, Fig. 473.

130. Without finding the traces, but in the plane of the lines AB and BC, Fig. 474, draw (*a*) an H-parallel AD; (*b*) a V-parallel CE; and (*c*) a P-parallel CF.

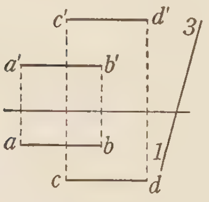

FIG. 475

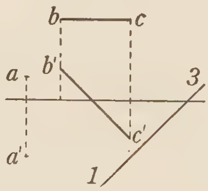

FIG. 476

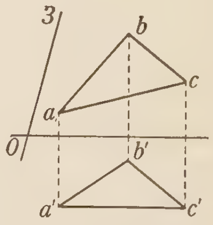

FIG. 477

131. Draw the lines AM and BN parallel to the coördinate plane 3 indicated and joining the lines AB and CD, Fig. 475. Project all four lines upon plane 3.

132. Draw the lines BM and CN parallel to the coördinate plane 3 indicated and in the plane of the point A and the line BC, Fig. 476. Project the point A and the three lines upon plane 3.

133. Draw the lines AD and BE parallel to the coördinate plane 3 indicated and in the plane of the triangle ABC, Fig. 477. Project AD, BE, and the triangle ABC upon plane 3.

Articles 35 and 36. Planes through a Line

134. Pass a plane T through any assigned line shown in Figs. 55 to 79, page 267.

Article 37. The Plane through Intersecting Lines

135. Pass a plane through any two assigned intersecting lines shown in Figs. 218 to 242, page 275.

136. Pass a plane R through the intersecting lines AC and BC, Fig. 478.

137. Find the plane of the triangle ABC, Fig. 479.

138. Find the plane of the triangle DEF, Fig. 480.

139. Show by construction whether the point O and the line DE are in the plane of the lines AB and BC, Fig. 481.

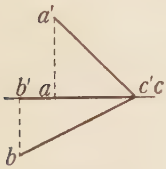

FIG. 478

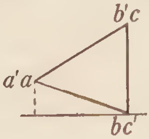

FIG. 479

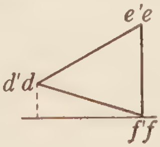

FIG. 480

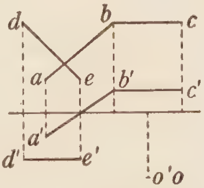

FIG. 481

140. Draw a 1-inch cube in the third quadrant with its faces parallel to the H, V, and P planes and $\frac{1}{2}''$ from each. Pass a plane through the intersecting solid diagonals running down right front and down left front.

141. Draw the cube of Problem 140. Pass a plane through the down left diagonal of the front face and the down rear diagonal of the right face. What third face diagonal is cut from the cube by the plane?

142. A 1-inch cube in the third quadrant has two faces parallel to the H plane. Two faces of the cube are inclined 30° and two are inclined 60° to the V plane. One corner of the cube is $\frac{1}{2}''$ from the H and V planes. Draw two traces of each of the six planes, Q, R, S, T, U, and W, that contain the faces of the cube.

143. A right pyramid in the third quadrant, with its axis 1″ long and perpendicular to the H plane, has a base which is 1″ square, with all its sides inclined 45° to the V plane. The apex is $\frac{1}{2}''$ from the H plane and 1″ from the V plane. Find the plane of the left front face.

144. Find the plane of the right rear face of the pyramid of Problem 143.

145. Draw a 1-inch cube in the first quadrant with one face parallel to the V plane and $\frac{1}{2}''$ from it, and with four faces inclined 45° to the H plane, the nearest edge being $\frac{1}{2}''$ from the H plane. Pass a plane through the upper left rear edge and the solid diagonal running down toward the front.

146. Pass a plane through any assigned plane figure shown in Figs. 323 to 338, page 280.

Article 38. The Plane through Two Parallel Lines

147. Draw a line OM through the given point O and parallel to the given line in any assigned set shown in Figs. 198 to 217, page 274, and then pass a plane through the two parallel lines.

148. Pass a plane through the two parallel lines AB and CD, Fig. 482.

149. Pass a plane through the two parallel lines EF and GJ, Fig. 483.

150. Pass a plane through the two parallel lines KL and MN, Fig. 484.

151. Pass a plane through the two parallel lines OP and QR, Fig. 485.

152. Pass a plane through the two parallel lines AB and CD, Fig. 486.

153. Pass a plane through the two parallel lines EF and GJ, Fig. 487.

154. Pass a plane through the two parallel lines KL and MN, Fig. 488.

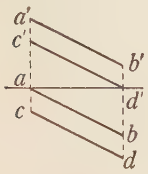

FIG. 482

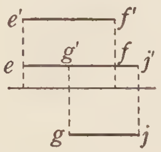

FIG. 483

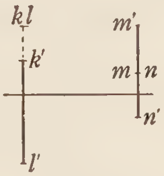

FIG. 484

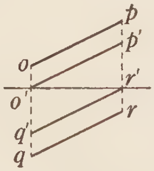

FIG. 485

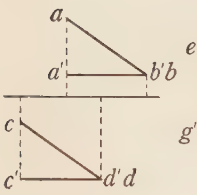

FIG. 486

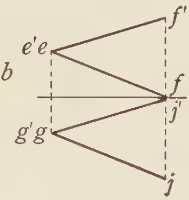

FIG. 487

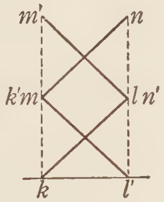

FIG. 488

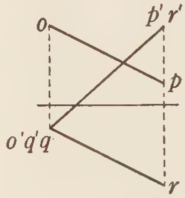

FIG. 489

155. Pass a plane through the two parallel lines OP and QR, Fig. 489.

Article 39. The Plane through a Point and a Line

156. Pass a plane through the point O and the given line in any assigned set shown in Figs. 198 to 217, page 274.

157. Pass a plane through the point O and the line AB, Fig. 490.

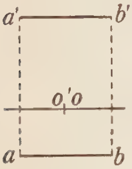

FIG. 490

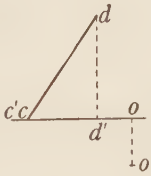

FIG. 491

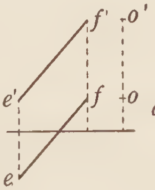

FIG. 492

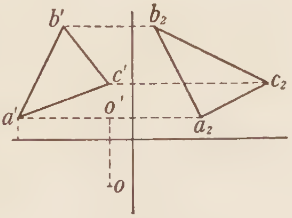

FIG. 493

158. Pass a plane through the point O and the line CD, Fig. 491.

159. Pass a plane through the point O and the line EF, Fig. 492.

160. Pass a plane Q through the triangle ABC, Fig. 493; a plane R through the point O and the line AB; a plane S through the point O and the line BC; and a plane T through the point O and the line CA. What geometrical figure is inclosed by the planes Q, R, S, and T?

Article 40. The Plane through Three Points

161. Pass a plane R through the three points A, B, and C, Fig. 494.

162. Pass a plane S through the three points E, F, and G, Fig. 495.

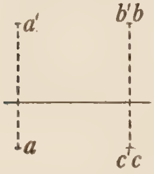

FIG. 494

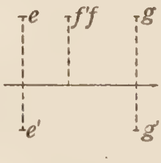

FIG. 495

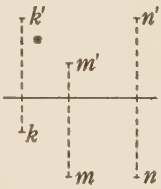

FIG. 496

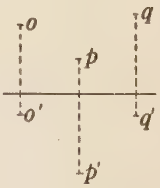

FIG. 497

163. Pass a plane T through the three points K, M, and N, Fig. 496.

164. Pass a plane X through the three points O, P, and Q, Fig. 497.

165. Pass a plane R through the three points A, B, and C, Fig. 498.

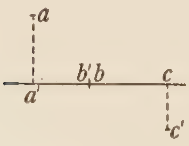

FIG. 498

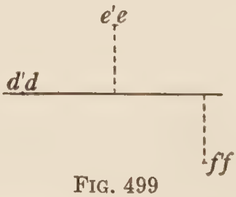

FIG. 499

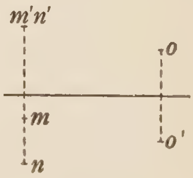

FIG. 500

166. Pass a plane S through the three points D, E, and F, Fig. 499.

167. Pass a plane T through the three points M, N, and O, Fig. 500.

168. Draw a 1-inch cube in the first quadrant with its faces parallel to the H, V, and P planes, the nearest faces being $\frac{1}{2}''$ from the planes. Pass a plane R through the upper left front corner, the lower left rear corner, and the midpoint of the right rear vertical edge.

169. Draw a 1-inch cube in the third quadrant with one face $\frac{1}{2}''$ from the H plane, the vertical faces inclined 45° to the V plane, and one edge $\frac{1}{2}''$ from the V plane. Pass a plane S through the upper front corner, the upper right corner, and the lower left corner.

170. Draw the cube of Problem 169. Pass a plane T through the upper left, the upper right, and the lower rear corners.

171. The points A, B, and C, Fig. 501, are on the outcrop of an ore vein. Find the plane and the strike of the vein. (The "strike" of a vein is the direction of a horizontal line in the plane of the vein).

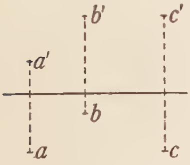

FIG. 501

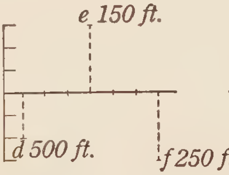

FIG. 502

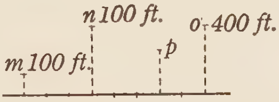

FIG. 503

172. Vertical holes drilled in a level field at the points D, E, and F meet a vein of ore at the depths indicated in Fig. 502. Find the plane and the strike of the vein. Scale: 1″ = 400 feet. (For definition of "strike," see Problem 171).

173. Test holes are drilled in a level field at the points M, N, and O, and meet a vein of ore at the depths indicated in Fig. 503. Find the plane of the vein. At what depth would a hole drilled at the point P meet the vein? Scale: 1″ = 400 feet.

174. Find the plane of the ellipse whose center is the point C, Fig. 504.

175. Find the plane of the ellipse whose center is the point D, Fig. 505.

176. Find the plane of the ellipse whose center is the point O, Fig. 506.

177. Find the plane that contains the curve of Fig. 507.

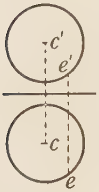

Fig. 504

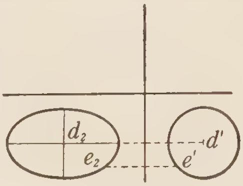

Fig. 505

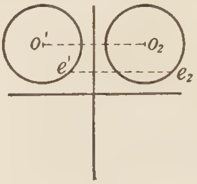

Fig. 506

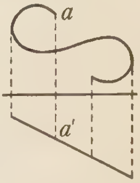

Fig. 507

Article 41. A Plane through a Point and Parallel to a Line

178. Pass a plane through the given point O and parallel to the given line in any assigned set shown in Figs. 198 to 217, page 274.

179. Pass a plane through the given point O and parallel to the given line AB, Fig. 508.

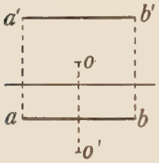

Fig. 508

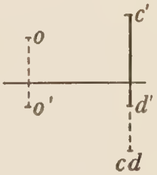

Fig. 509

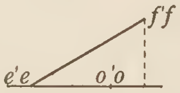

Fig. 510

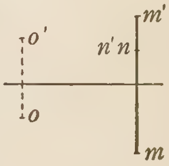

Fig. 511

180. Pass a plane through the given point O and parallel to the given line CD, Fig. 509.

181. Pass a plane through the given point O and parallel to the given line EF, Fig. 510.

182. Pass a plane through the given point O and parallel to the given line MN, Fig. 511.

183. Through the given point O, Fig. 512, draw a line OM parallel to the given plane R.

184. Through the given point O, Fig. 513, draw a line OM parallel to the given plane S.

185. Through the given point O, Fig. 514, draw a line OM parallel to the given plane T.

186. Through the given point O, Fig. 515, draw a line OM parallel to the given plane X.

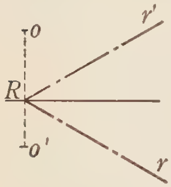

FIG. 512

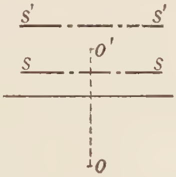

FIG. 513

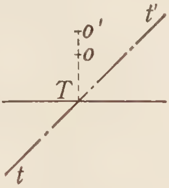

FIG. 514

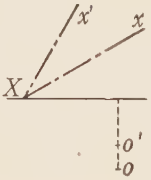

FIG. 515

187. The line AB, Fig. 516, is parallel to the given plane R. Find the V trace of R.

188. The line CD, Fig. 517, is parallel to the given plane S. Find the H trace of S.

189. The line EF, Fig. 518, is parallel to the given plane T. Find the H trace of T.

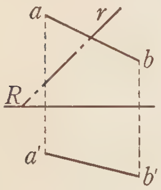

FIG. 516

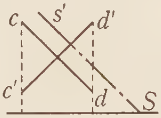

FIG. 517

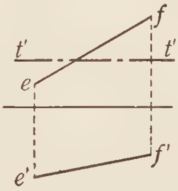

FIG. 518

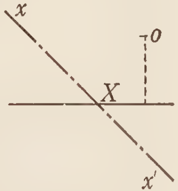

FIG. 519

190. Draw a line OM in the given plane X, Fig. 519, and parallel to the P plane.

191. The line AB, Fig. 520, is parallel to the given plane R. Find the V trace of R. How many solutions are there?

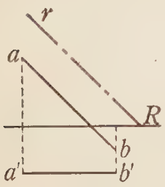

FIG. 520

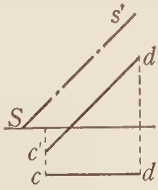

FIG. 521

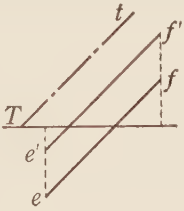

FIG. 522

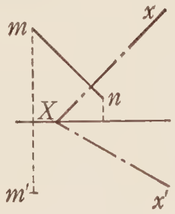

FIG. 523

192. The line CD, Fig. 521, is parallel to the given plane S. Find the H trace of S. How many solutions are there?

193. The line EF, Fig. 522, is parallel to the given plane T. Find the V trace of T. How many solutions are there?

194. The line MN, Fig. 523, is parallel to the given plane X. Find the V projection of MN. How many solutions are there?

Article 42. The Plane through One Line and Parallel to Another Line

195. Pass a plane through the line AB and parallel to the second given line in any assigned set shown in Figs. 243 to 262, page 276.

196. Pass a plane R through the line AB, Fig. 524, and parallel to the horizontal ground line.

197. Pass a plane S through the horizontal ground line and parallel to the given line AB, Fig. 524.

198. Pass a plane T through the line AB, Fig. 525, and parallel to the given line CD.

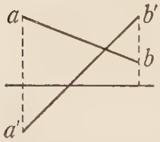

FIG. 524

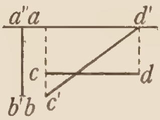

FIG. 525

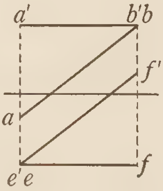

FIG. 526

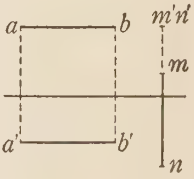

FIG. 527

199. Pass a plane U through the line AB, Fig. 526, and parallel to the given line EF.

200. Pass a plane X through the line AB, Fig. 527, and parallel to the given line MN.

201. Pass a plane R through the line AB, Fig. 528, and parallel to the given line CD.

202. Pass a plane S through the line AB, Fig. 529, and parallel to the given line EF.

203. Pass a plane T through the line AB, Fig. 530, and parallel to the given line MN.

204. Pass a plane X through the line AB, Fig. 531, and parallel to the given line OP.

205. Pass a plane R through the line AB, Fig. 532, and parallel to the given line CD.

206. Pass a plane S through the line AB, Fig. 533, and parallel to the given line EF.

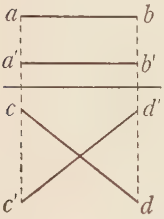

FIG. 528

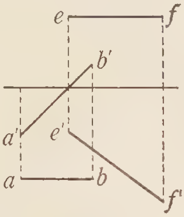

FIG. 529

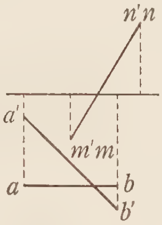

FIG. 530

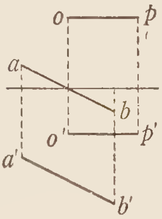

FIG. 531

207. Pass a plane T through the line AB, Fig. 534, and parallel to the given line MN.

208. Pass a plane X through the line AB, Fig. 535, and parallel to the given line OP.

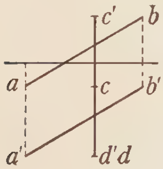

FIG. 532

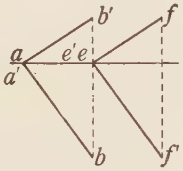

FIG. 533

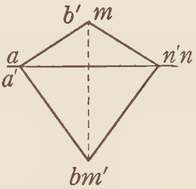

FIG. 534

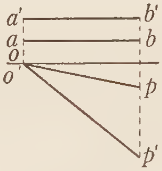

FIG. 535

Article 43. The Plane through a Point and Parallel to Two Lines

209. Select a point O approximately 1″ to the right of the point B (full scale) and as far from the H and V planes as point B is, and pass a plane through O parallel to the two given lines in any assigned set shown in Figs. 243 to 262, page 276.

210. Pass a plane R through the point O and parallel to the lines AB and CD, Fig. 536.

211. Pass a plane S through the point O and parallel to the lines EF and FG, Fig. 537.

212. Pass a plane T through the point O and parallel to the lines MN and MP, Fig. 538.

213. Pass a plane R through the point P and parallel to the lines AB and AC, Fig. 539.

214. Pass a plane S through the point P and parallel to the lines DE and FG, Fig. 540.

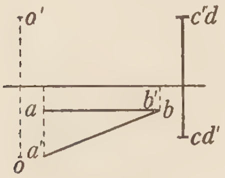

FIG. 536

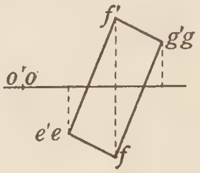

FIG. 537

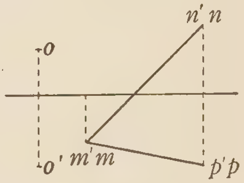

FIG. 538

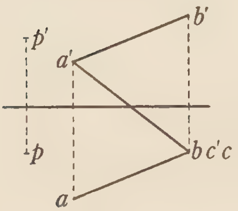

FIG. 539

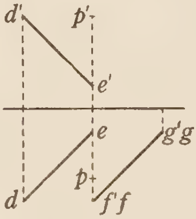

FIG. 540

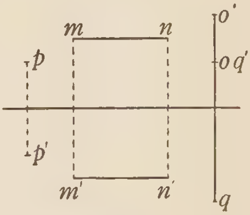

FIG. 541

215. Pass a plane T through the point P and parallel to the lines MN and OQ, Fig. 541.

Article 44. The Plane through a Point and Parallel to a Plane

216. Pass a plane through the given point O and parallel to the given plane in any assigned set shown in Figs. 263 to 282, page 277.

217. Pass a V projecting plane R through the line AB, Fig. 542. Pass a plane S through the point O and parallel to the plane R.

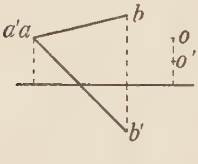

FIG. 542

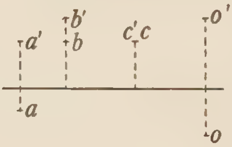

FIG. 543

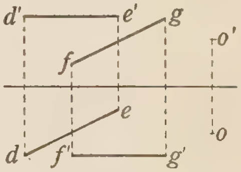

FIG. 544

218. Pass a plane S through the point O, Fig. 543, and equidistant from the points A, B, and C.

219. Pass a plane T through the point O, Fig. 544, and equidistant from the lines DE and FG.

Article 45. Lines Perpendicular to Planes

220. Draw a line OM through the given point O and perpendicular to the given plane in any assigned set shown in Figs. 263 to 282, page 277.

221. The line AB, Fig. 545, is perpendicular to the plane T. Draw the V projection of AB and the H trace of T.

222. The line CD, Fig. 546, is contained in the plane R, is parallel to the plane S, and perpendicular to the plane T. Find the H traces of R, S, and T, and the V projection of CD.

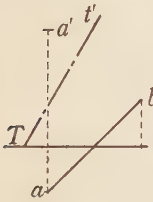

FIG. 545

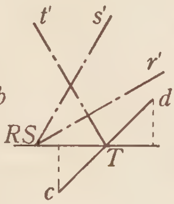

FIG. 546

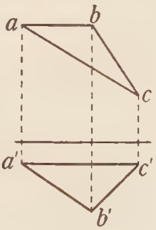

FIG. 547

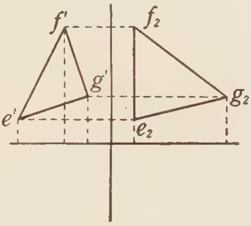

FIG. 548

223. Draw a line AK perpendicular to the plane of the triangle ABC, Fig. 547.

224. Draw a line EM, of any length, perpendicular to the plane of the triangle EFG, Fig. 548. Draw lines FN and GO parallel to and equal in length to the line EM. Connect the six points in the proper order to form the triangular prism EFG–MNO.

Article 47. A Plane through a Point and Perpendicular to a Plane

225. Pass a plane M through the given point O and perpendicular to the given plane in any assigned set shown in Figs. 263 to 282, page 277.

226. Set up an auxiliary coördinate plane 3 through the given point O and perpendicular to the given plane in any assigned set shown in Figs. 263 to 282, page 277, and then find the auxiliary trace of the given plane.

Article 48. A Plane through a Line and Perpendicular to a Plane

227. Pass a plane M through the given line and perpendicular to the given plane in any assigned set shown in Figs. 283 to 302, page 278.

228. Pass a plane R through the point O, Fig. 549, parallel to the line AB and perpendicular to the plane S.

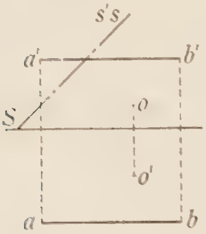

FIG. 549

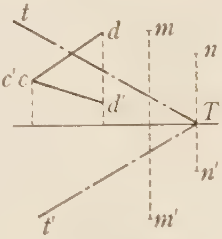

FIG. 550

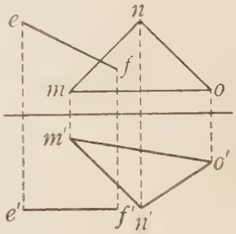

FIG. 551

229. Pass a plane R through the point O taken midway between the points M and N, Fig. 550, parallel to the line CD and perpendicular to the plane T.

230. Pass a plane R through the line EF, Fig. 551, and perpendicular to the plane of the triangle MNO.

Article 49. A Plane through a Point and Perpendicular to Two Planes

231. Pass a plane M through any chosen point A and perpendicular to the two given planes in any assigned set shown in Figs. 303 to 322, page 279.

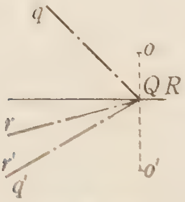

FIG. 552

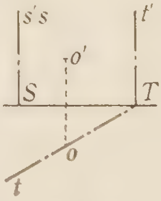

FIG. 553

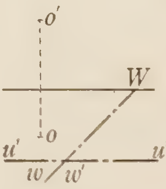

FIG. 554

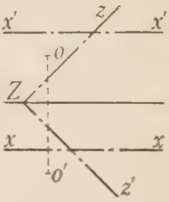

FIG. 555

232. Pass a plane N through the given point O and perpendicular to the two given planes in any assigned set shown in Figs. 552 to 563.

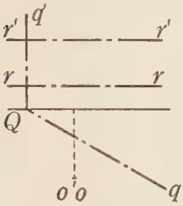

FIG. 556

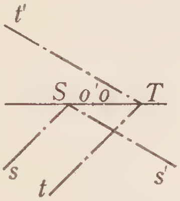

FIG. 557

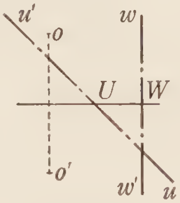

FIG. 558

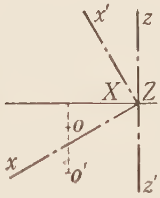

FIG. 559

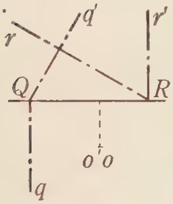

FIG. 560

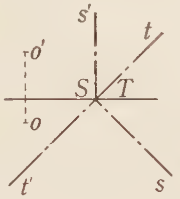

FIG. 561

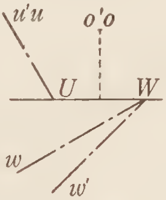

FIG. 562

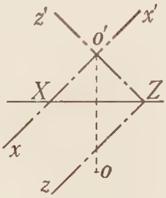

FIG. 563

233. How must a plane be passed so that it will cut, from two given planes, lines which measure the dihedral angle between the given planes?

Article 50. A Plane through a Point and Perpendicular to a Line

234. Pass a plane R through the given point O and perpendicular to the given line in any assigned set shown in Figs. 198 to 217, page 274.

235. Find the locus of a point M equidistant from the given points A and B, Fig. 564.

236. Find the locus of a point M equidistant from the given points C and D, Fig. 565.

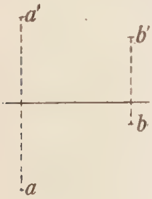

FIG. 564

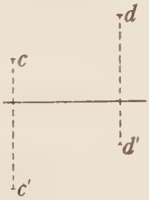

FIG. 565

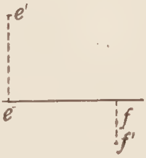

FIG. 566

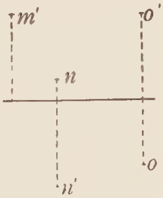

FIG. 567

237. Find the locus of a point M equidistant from the given points E and F, Fig. 566.

238. The point M is equidistant from the given points N and O, Fig. 567. Find the H projection of M.

239. The line AV, Fig. 568, is the base of an isosceles triangle ABV. Find the locus of the vertex, B.

240. The line CE, Fig. 569, is the base of an isosceles triangle CDE, whose vertex, D, is in the H plane. Find the locus of D.

241. The line FG, Fig. 570, is the base of an isosceles triangle FKG, whose vertex, K, is in the ground line. Draw the triangle FKG.

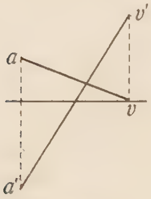

FIG. 568

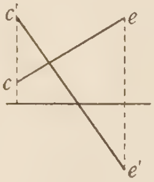

FIG. 569

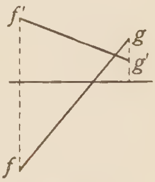

FIG. 570

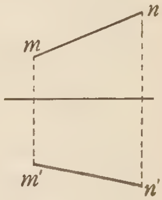

FIG. 571

242. The line MN, Fig. 571, is the base of an isosceles triangle MON, whose vertex, O, is in the ground line. Draw the triangle MON.

243. The angle ABC, Fig. 572, is a right angle. Find the H projection of the line BC.

244. The line CD, in the plane R, Fig. 573, is the base of an isosceles triangle CDE. Construct the triangle.

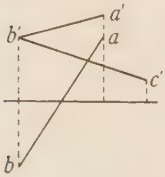

FIG. 572

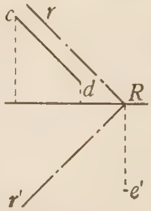

FIG. 573

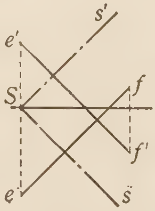

FIG. 574

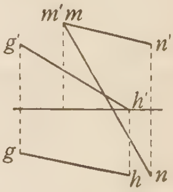

FIG. 575

245. Draw a line FG that is parallel to the plane S, Fig. 574, and perpendicular to the line EF.

246. The line MN, Fig. 575, is the base of an isosceles triangle MON, of which the vertex, O, lies in the line GH. Construct the triangle MON.

Article 51. The Line of Intersection of Two Planes

247. Find the line of intersection of the two given planes in any assigned set shown in Figs. 303 to 322, page 279.

248. Find the locus of a point A which lies in the plane R, Fig. 576, and is 1″ above the H plane.

249. The planes S and T, Fig. 577, intersect in the line AB. Find the missing projection and traces.

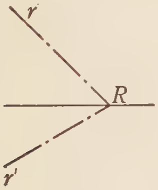

FIG. 576

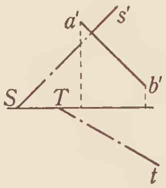

FIG. 577

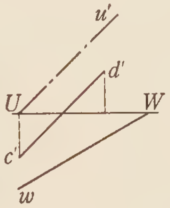

FIG. 578

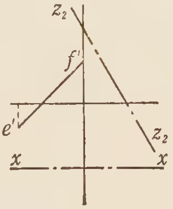

FIG. 579

250. The planes U and W, Fig. 578, intersect in the line CD. Find the missing projection and traces.

251. The planes X and Z, Fig. 579, intersect in the line EF. Locate the missing projection and traces.

252. Find the locus of a point O that is equidistant from the points A and B, Fig. 580, and in the plane R.

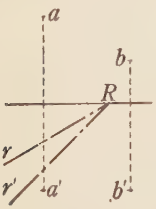

FIG. 580

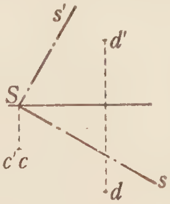

FIG. 581

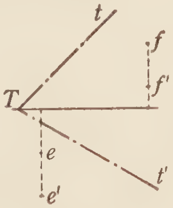

FIG. 582

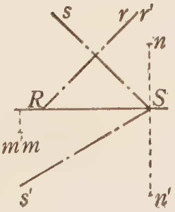

FIG. 583

253. Find the locus of a point O that is equidistant from the points C and D, Fig. 581, and in the plane S.

254. Find the locus of a point O that is equidistant from the points E and F, Fig. 582, and in the plane T.

255. Find the plane T that is perpendicular to the planes R and S, Fig. 583, and equidistant from the points M and N.

256. Draw the line BO perpendicular to the line AB and parallel to the plane R, Fig. 584.

257. Draw the line DO perpendicular to the line CD and parallel to the plane S, Fig. 585.

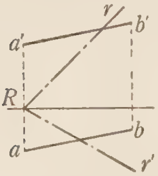

FIG. 584

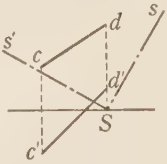

FIG. 585

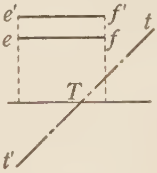

FIG. 586

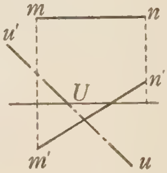

FIG. 587

258. Draw the line FO perpendicular to the line EF and parallel to the plane T, Fig. 586.

259. Draw the line NO perpendicular to the line MN and parallel to the plane U, Fig. 587.

Article 52. The Point in which a Line Pierces a Plane

260. Find the point P in which the given line pierces the given plane in any assigned set shown in Figs. 283 to 302, page 278. Use the projecting-plane method.

261. Find the point P in which the given line pierces the given plane in any assigned set shown in Figs. 283 to 302, page 278. Use the auxiliary-coördinate-plane method.

262. Find the point P in which the given line pierces the plane of the given polygon in any assigned set shown in Figs. 339 to 354, page 281, without finding the traces of the plane.

263. Find the point P, in the line AB, that is equidistant from the points C and D, Fig. 588.

264. Find the point P, in the line AB, that is equidistant from the points E and F, Fig. 589.

265. Find the point P, in the line AB, equidistant from the points G and K, Fig. 590.

266. Find the point P, in the line AB, equidistant from the points M and N, Fig. 591.

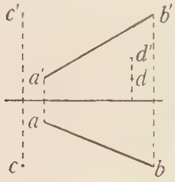

FIG. 588

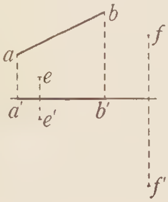

FIG. 589

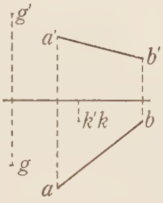

FIG. 590

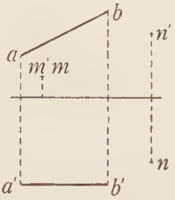

FIG. 591

267. Draw the right triangle ABC whose vertex, B, is in the line DE, Fig. 592, with the right angle at the point A.

268. Draw the right triangle ABC whose vertex, B, is in the line FG, Fig. 593, with the right angle at the point C.

269. Draw the isosceles triangle ABC whose base is the line AB, Fig. 594, and whose vertex, C, is in the line MN.

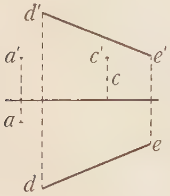

FIG. 592

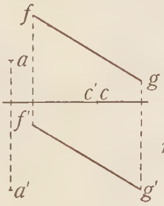

FIG. 593

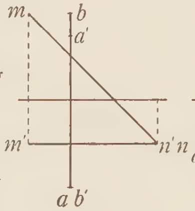

FIG. 594

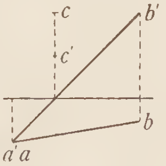

FIG. 595

270. Draw the isosceles triangle BCD whose base, BD, is in the line AB, Fig. 595, and whose vertex is the point C.

271. Locate the point M that is equidistant from the points A and B, Fig. 596, and in the planes R and S.

272. Locate the point M that is equidistant from the points C and D, Fig. 597, and in the planes T and U.

273. Locate the point M that is equidistant from the points E and F, Fig. 598, and in the planes X and Z.

274. Locate the point M that is equidistant from the points A, B, and C, Fig. 599, and in the plane R.

275. Find the point P in which the line of intersection of the planes R and S, Fig. 600, pierces the plane T.

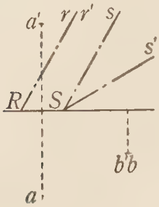

FIG. 596

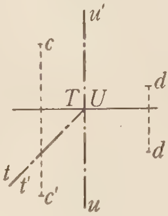

FIG. 597

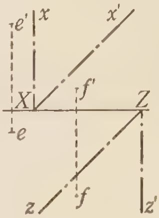

FIG. 598

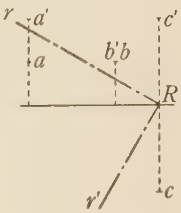

FIG. 599

276. Find the point P in which the line MN, Fig. 601, pierces the plane of the triangle ABC. Do not draw the traces of the plane of the triangle.

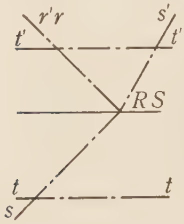

FIG. 600

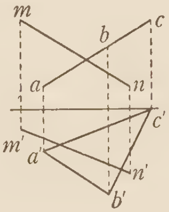

FIG. 601

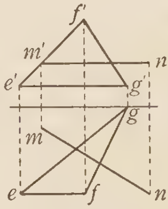

FIG. 602

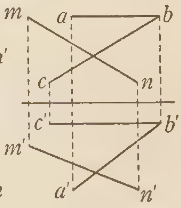

FIG. 603

277. Find the point P in which the line MN, Fig. 602, pierces the plane of the triangle EFG. Do not draw the traces of the plane of the triangle.

278. Find the point P in which the line MN, Fig. 603, pierces the plane of the lines AB and BC. Do not draw the traces of the plane of AB and BC.

Article 53. The Projection of a Line upon an Oblique Plane

279. Find the projection of the given line upon the given plane in any assigned set shown in Figs. 283 to 302, page 278. Use the projecting-plane method.

280. Find the projection of the given line upon the given plane in any assigned set shown in Figs. 283 to 302, page 278. Use the auxiliary-coördinate-plane method.

281. Find the projection of the given line upon the plane of the given polygon in any assigned set shown in Figs. 339 to 354, page 281.

Article 54. The Revolution of an Oblique Plane

282. Pass any plane not an H or V projecting plane through any assigned line shown in Figs. 55 to 79, page 267, and then revolve the plane about one of its traces into the corresponding coördinate plane, and show the given line in its revolved position. Mark this the true length (T. L.) of the line.

283. Pass a plane through any assigned polygon shown in Figs. 323 to 338, page 280, and then revolve the plane about one of its traces into the corresponding coördinate plane, and show the polygon in its revolved position. Mark this the true size (T. S.) of the polygon.

Article 55. The Counter-revolution of an Oblique Plane; Construction of Plane Geometrical Figures

284. The point C, Fig. 604, is the center of a circle x'' in diameter in the plane R. Draw the projections of the circle.

285. The point b_r, Fig. 605, is the revolved position of the vertex of an isosceles triangle ABC in the plane S. The base, AC, is parallel to the H plane and equal in length to the altitude of the triangle, which is x'' long. Draw the projections of one of the four possible triangles.

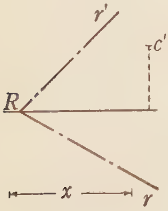

FIG. 604

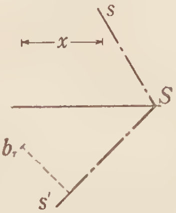

FIG. 605

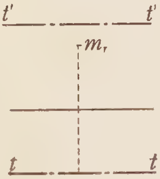

FIG. 606

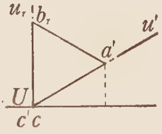

FIG. 607

286. The point b_r, Fig. 605, is the revolved position of one corner of a square ABCD in the plane S. The square is x'' on a side, with two sides parallel to the V plane. Draw the projections of one of the eight possible squares.

287. The point b_r, Fig. 605, is the revolved position of the center of a square in the plane S. The square is x'' on a side, with one diagonal parallel to the V plane. Draw the projections of one of the two possible squares.

288. The point M, in the plane T, Fig. 606, falls at m_r when T is revolved into the H plane about tt as an axis. Find the H and V projections of M.

289. Counter-revolve the triangle ABC, Fig. 607, about the V trace, Uu', of the plane U until the side Uu_r is in the H plane.

Articles 57 to 60. The Auxiliary-Coördinate-Plane-4 Projection of a Point or a Line

290. Project any assigned point shown in Figs. 16 to 30, page 265, upon an auxiliary coördinate plane 4 set up with its ground line, 3–4, in any chosen position with respect to the plane-3 projection of the given point.

291. Project any assigned line shown in Figs. 80 to 103, page 268, upon an auxiliary coördinate plane 4 set up with its ground line, 3–4, inclined at any angle to the plane-3 projection of the given line.

292. The line AB, Fig. 608, lies in the third quadrant and to the left of plane 3. Find the H and V projections of the line.

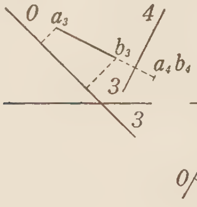

FIG. 608

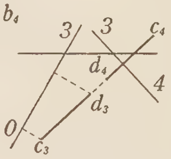

FIG. 609

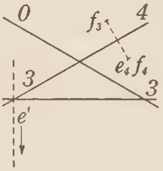

FIG. 610

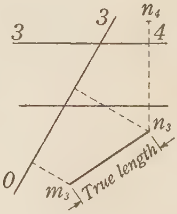

FIG. 611

293. The line CD, Fig. 609, lies in the first quadrant, with the point C to the left of plane 3. Find the H and V projections of CD.

294. The line EF, Fig. 610, lies in the third quadrant and to the left of plane 3. Find its H, V, and plane-3 projections.

295. Complete the plane-4 projection and find the H and V projections of the line MN, Fig. 611.

Article 61. The Auxiliary Coördinate Plane Perpendicular to a Line

296. Find the projections of any assigned line shown in Figs. 55 to 79, page 267, upon an auxiliary coördinate plane set up perpendicular to the given line.

297. Project any assigned plane surface shown in Figs. 323 to 338, page 280, upon an auxiliary coördinate plane set up perpendicular to any chosen line in the surface. This gives the straight-line projection, or edgewise view, of the surface.

Article 62. The Auxiliary Coördinate Plane Parallel to the Plane of Two Lines

298. Project any assigned set of intersecting lines shown in Figs. 218 to 242, page 275, upon an auxiliary coördinate plane set up parallel to the plane of the two given lines.

299. Project (find the true size of) any assigned plane surface shown in Figs. 323 to 338, page 280, upon an auxiliary coördinate plane set up parallel to the plane of the given surface.

Article 63. The True Length of a Line

300. Find the true length of any assigned line shown in Figs. 55, 57, 60, 61, 62, 63, 67, 68, 69, 70, 71, 76, 78, 79, page 267, by the method of (*a*) revolving the line about a projector of one of its ends, and (*b*) revolving the line about one of its projections.

Article 64. The Shortest Distance from a Point to a Plane

301. Find the shortest distance from the given point O to the given plane in any assigned set shown in Figs. 263 to 282, page 277.

302. Find the shortest distance from an assumed point O to any assigned plane surface shown in Figs. 323 to 338, page 280, by using an auxiliary coördinate plane, and without drawing the traces of the plane of the given surface.

303. What is the size of a cube which has one corner at the point A, Fig. 612, and one face in the plane R?

304. What is the altitude of an oblique pyramid which has its apex at the point B, Fig. 613, and its base in the plane S?

305. A sphere whose center is at the point C, Fig. 614, is tangent to the plane T. Find the radius of the sphere and its point of tangency with the plane.

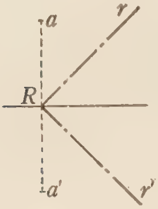

FIG. 612

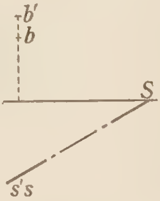

FIG. 613

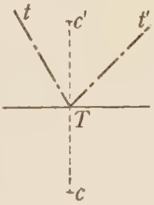

FIG. 614

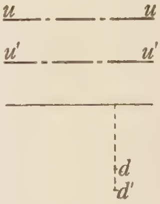

FIG. 615

306. The point D, Fig. 615, is the apex of a right circular cone whose base lies in the plane U. Find the altitude of the cone and the center, C, of the base of the cone.

307. The point O is in the plane R, Fig. 616. Draw a line OM perpendicular to R and 1″ long.

308. Determine the point B which is as far from the plane S, Fig. 617, as the point A and in the same line perpendicular to plane S.

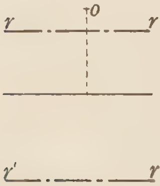

FIG. 616

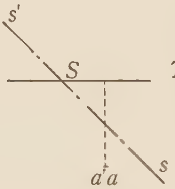

FIG. 617

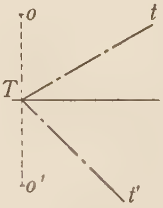

FIG. 618

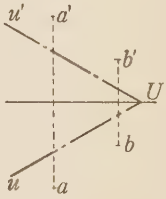

FIG. 619

309. Assuming the plane T, Fig. 618, to be a mirror, find the image of the point O in the mirror.

310. A ray of light passes through the point A, Fig. 619, reflects from the plane mirror U at a point O, and passes through the point B. Locate the point O.

Article 65. The Shortest Distance between Two Parallel Planes

311. Find the shortest distance between the two parallel planes Q and R, Fig. 620.

312. Find the shortest distance between the two parallel planes S and T, Fig. 621.

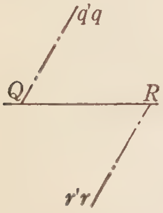

FIG. 620

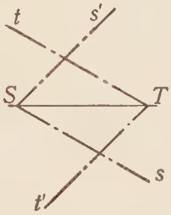

FIG. 621

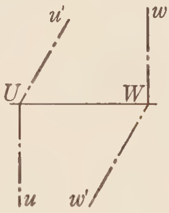

FIG. 622

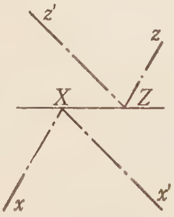

FIG. 623

313. Find the shortest distance between the two parallel planes U and W, Fig. 622.

314. Find the shortest distance between the two parallel planes X and Z, Fig. 623.

315. The planes Q and R, Fig. 624, are parallel and a distance *d* apart. Find their V traces.

316. Find the distance between the parallel planes R and S, Fig. 625.

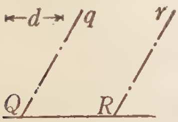

FIG. 624

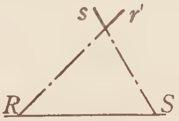

FIG. 625

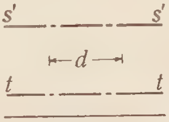

FIG. 626

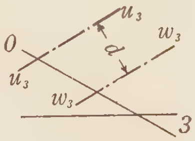

FIG. 627

317. The planes S and T, Fig. 626, are parallel and a distance *d* apart. Find the H trace of S and the V trace of T.

318. The planes U and W, Fig. 627, are a distance *d* apart. Find the H and V traces of the two planes.

319. An oblique pyramid of 1-inch altitude has a square base in the plane R, Fig. 628, with the center at C. Draw the locus of the apex of the pyramid.

320. A 1-inch cube has one face in the plane S, Fig. 629. Find the plane Q which contains the opposite face.

321. A sphere of 1-inch radius is tangent to the plane T, Fig. 630. Find the locus of the center of the sphere.

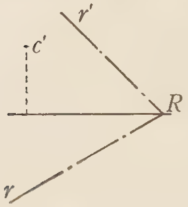

Fig. 628

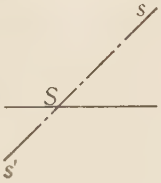

Fig. 629

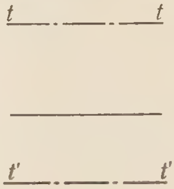

Fig. 630

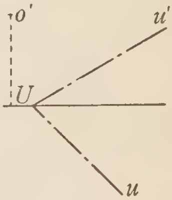

Fig. 631

322. The point O is 1″ from the plane U, Fig. 631. Find the V projection of the point.

Article 66. The Shortest Line from a Point to a Line

323. Find the projections and true length of the shortest line from the given point O to the given line in any assigned set shown in Figs. 198 to 217, page 274. Use the auxiliary-coördinate-plane method to check the oblique-plane result, and vice versa.

324. Locate the point D in the line AB, Fig. 632, so that the triangle BCD will have a right angle at D.

325. Find the center, C, and the radius of motion if the point A is revolved about the line EF, Fig. 633.

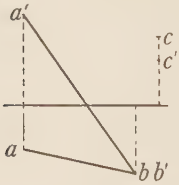

Fig. 632

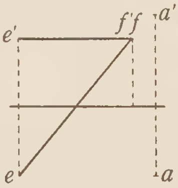

Fig. 633

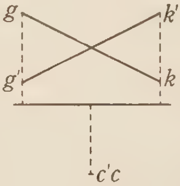

Fig. 634

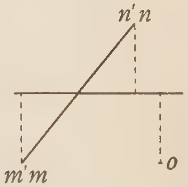

Fig. 635

326. Find the radius, CD, of a circle tangent to the line GK, Fig. 634, with its center at the point C.

327. The point O, Fig. 635, is 2″ from the line MN. Find the V projection of O.

Article 67. The Shortest Distance between Two Parallel Lines

328. Find the shortest distance between the two parallel lines AB and CD, Fig. 636.

329. Find the shortest distance between the two parallel lines AB and EF, Fig. 637.

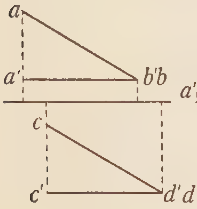

FIG. 636

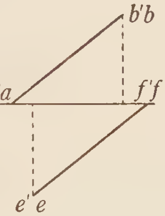

FIG. 637

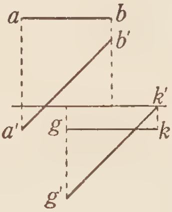

FIG. 638

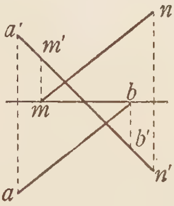

FIG. 639

330. Find the shortest distance between the two parallel lines AB and GK, Fig. 638.

331. Find the shortest distance between the two parallel lines AB and MN, Fig. 639.

332. The lines AB and CD, Fig. 640, are parallel and a distance *d* apart. Find the H projection of CD.

333. The lines AB and EF, Fig. 641, are parallel, of equal length, and a distance *d* apart. Find the H projection of AB and the V projection of EF.

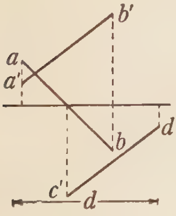

FIG. 640

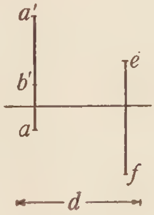

FIG. 641

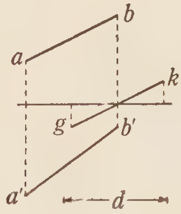

FIG. 642

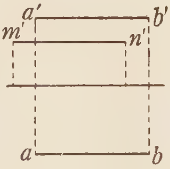

FIG. 643

334. The lines AB and GK, Fig. 642, are parallel and a distance *d* apart. Find the V projection of GK.

335. The line MN, Fig. 643, is in the second quadrant and as far from the line AB as AB is from the horizontal ground line. Find the H projection of MN.

Articles 68 and 69. The Angle between Two Intersecting Lines, and its Bisector

336. Find the true size of the angle X between any two assigned intersecting lines shown in Figs. 218 to 242, page 275. Use the auxiliary-coördinate-plane method to check the oblique-plane result, and vice versa.

337. Find the projections of the bisector of the angle X between any two assigned intersecting lines shown in Figs. 218 to 242, page 275.

338. Locate the center, C, of the circle that may be inscribed in any assigned triangle shown in Figs. 323 to 331, page 280.

Article 70. A Line through a Point making a Specified Angle with a Line

339. Draw a line OM through the given point O and making a specified angle X with the given line in any assigned set shown in Figs. 198 to 217, page 274. Use the auxiliary-coördinate-plane method to check the oblique-plane result, and vice versa.

Article 71. The Angle between a Line and a Plane

340. Find the acute angle X between the given line and the given plane in any assigned set shown in Figs. 283 to 302, page 278.

DEFINITION. The angle of incidence of a ray of light striking a plane mirror is the angle between the ray and the normal to the mirror at the point of incidence. The angle of reflection equals the angle of incidence, and is defined as the angle between the normal and the reflected ray. (See Problems 341 to 348.)

341. The line AB, Fig. 644, represents a ray of light. Find its path after being reflected from the H plane. Find the angle of incidence.

342. The line CD, Fig. 645, represents a ray of light. Find its path after being reflected from the V plane. Find the angle of incidence.

343. Find the path of the ray of light EF, Fig. 646, after being reflected from both the H and the V planes in order. Find both angles of incidence.

344. Find the path of the ray of light MN, Fig. 647, after being reflected from the H, P, and V planes in order. Find all three angles of incidence.

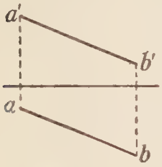

FIG. 644

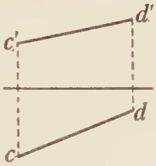

FIG. 645

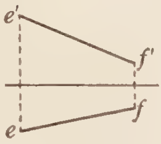

FIG. 646

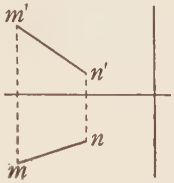

FIG. 647

345. Find the path of the ray of light AB, Fig. 648, after it is reflected by the plane mirror S. Find the angle of incidence.

346. Find the point B in the V plane at which a ray of light through the point A, Fig. 649, is reflected by the plane to pass through the point C.

347. Find the point D in the H plane at which a ray of light through the point C, Fig. 650, is reflected by the plane to pass through the point E.

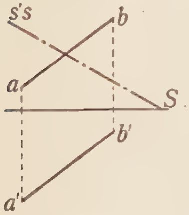

FIG. 648

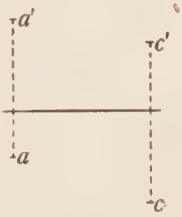

FIG. 649

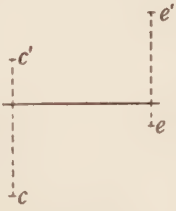

FIG. 650

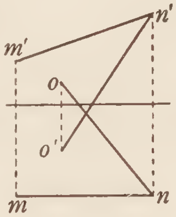

FIG. 651

348. Find the traces of a plane mirror T passed through the point N, Fig. 651, in such a way that the ray of light MN reflects along the path NO.

Articles 72 and 73. The Angle between Two Intersecting Planes, and its Plane-Bisector

349. Find the acute angle X between the two given planes in any assigned set shown in Figs. 303 to 322, page 279.

350. Find the plane bisector of either angle between the two given planes in any assigned set shown in Figs. 303 to 322, page 279.

351. Find the locus of a point equidistant from the planes Q and R, Fig. 652.

352. Locate the point C, in the line AB, Fig. 653, that is the center of a sphere tangent to the planes S and T.

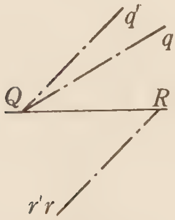

FIG. 652

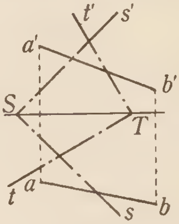

FIG. 653

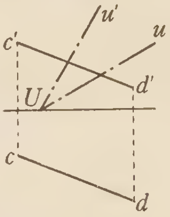

FIG. 654

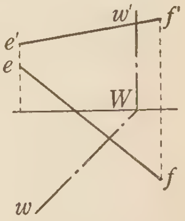

FIG. 655

353. Draw a sphere tangent to the plane U and to the H plane, Fig. 654, and having its center in the line CD.

354. A sphere whose center is in the line EF, Fig. 655, is tangent to the plane W and to the H plane. Locate the center of the sphere.

Article 74. A Plane through a Line and making a Specified Angle with a Given Plane

355. Assume a line in any assigned plane shown in Figs. 133 to 157, page 271, and through this line pass a plane which makes any specified angle X with the given plane.

Articles 75 and 76. The Angles Θ, Φ, and Π of a Line

356. Find the angles Θ, Φ, and Π for any assigned line shown in Figs. 55 to 79, page 267.

Article 77. The Perpendicular Line between Two Non-Parallel, Non-Intersecting Lines

357. Find the true length of the common perpendicular between the two given lines in any assigned set shown in Figs. 243 to 262, page 276, by use of auxiliary coördinate planes.

Articles 78 and 79. The Construction of Right Pyramids and Cones

358. Draw a right square pyramid with its apex at the given point O and its base, with sides equal in length to the altitude of the pyramid, lying in the given plane in any assigned set shown in Figs. 263 to 282, page 277. Use the oblique-plane method.

359. Draw a right circular cone with its apex at the given point O and its base, with diameter equal to the altitude of the cone, lying in the given plane in any assigned set shown in Figs. 263 to 282, page 277. Use the auxiliary-coördinate-plane method.

360. Draw a pyramid with any assigned triangle shown in Figs. 323 to 331, page 280, as the base, and with an altitude, of any chosen length, perpendicular to the base at the center of the circle inscribed in the triangle.

361. The point A, Fig. 656, is the apex of a right square pyramid whose base lies in the plane R. A diagonal of the base is parallel to the H plane, and the sides of the base are equal in length to the altitude of the pyramid. Draw the projections of the pyramid.

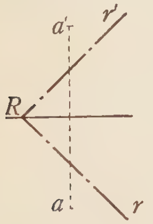

FIG. 656

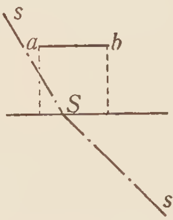

FIG. 657

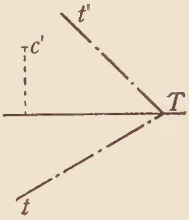

FIG. 658

362. The line AB, Fig. 657, is one diagonal of a square contained in the plane S. Construct a right pyramid with the square as its base and its altitude equal to AB.

363. The point C, Fig. 658, is the center of the base of a right circular cone. The circular base is in the plane T. Draw the projections of the cone with the altitude and diameter of base $1\frac{1}{2}''$ in length.

Article 80. The Construction of Right Prisms and Cylinders

364. Construct a right prism of any convenient altitude with any assigned polygon shown in Figs. 323 to 338, page 280, as the base. Use the auxiliary-coördinate-plane method of solution.

Articles 81 and 82. Tangents and Normals to Plane Curves

365. Draw a line AB and a line CD tangent and normal, respectively, at any point O, to a circle or an ellipse drawn in any assigned plane shown in Figs. 133 to 157, page 271.

Articles 83 to 91. The Construction of the Fundamental Elements of Surfaces

366. Given the oblique cone A, Fig. 659, locate (*a*) an element AN on the upper back portion of the cone; (*b*) a point E on the lower front portion of the cone; (*c*) a point F on the back portion of the base; and (*d*) the plane R of the base.

367. Extend the cone A, Fig. 659, until it intersects the horizontal plane, and show the H and V projections of this new base. Find the true size of the given base.

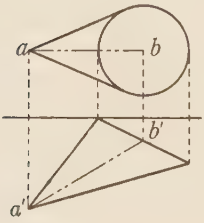

FIG. 659

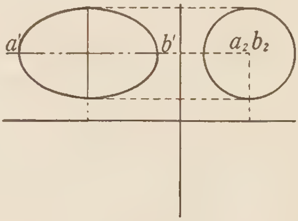

FIG. 660

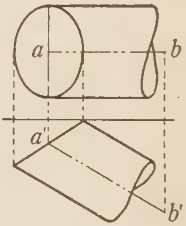

FIG. 661

368. Given the prolate ellipsoid of revolution whose axis is AB, Fig. 660, locate (*a*) a point D on the upper back portion of the ellipsoid, and (*b*) a point E on the lower front portion of the ellipsoid.

369. Given the oblique cylinder whose axis is AB, Fig. 661, locate (*a*) an element MN on the lower front portion of the cylinder; (*b*) a point D on the upper back portion of the cylinder; (*c*) a point E on the base; and (*d*) the plane R of the base.

370. Extend the cylinder whose axis is AB, Fig. 661, to the H plane and show the H and V projections of this new base. Find the true size of the given base.

Articles 92 and 93. The Line of Intersection of Two Plane Polygons

371. Find the line of intersection, MN, of the two given polygons in any assigned set shown in Figs. 662 to 667. Consider the polygons opaque and show all invisible portions by broken lines.

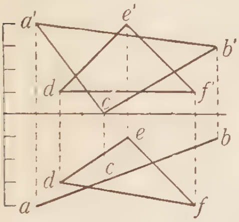

FIG. 662

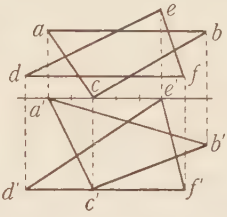

FIG. 663

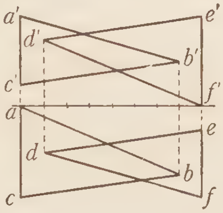

FIG. 664

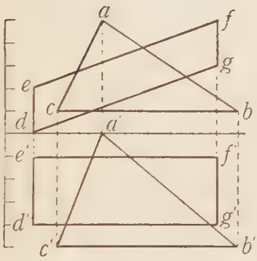

FIG. 665

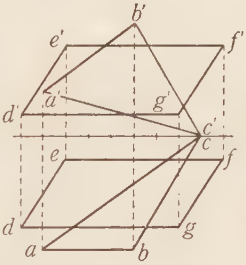

FIG. 666

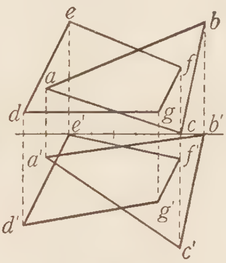

FIG. 667

Articles 95 to 98. The Line of Intersection of a Plane and a Curved Surface

372. Find the line of intersection of the given plane and the given curved surface in any assigned set shown in Figs. 355 to 366, page 282. Show the invisible portions of the intersection by a broken line.

Articles 99 to 102. The Line of Intersection of Two Curved Surfaces

373. Find the line of intersection of the two given curved surfaces in any assigned set shown in Figs. 367 to 375, page 283. Show invisible portions of the intersection by the usual broken line.

Articles 103 to 106. The Development (Pattern) of a Cone

374. Develop any assigned cone shown in Figs. 668 to 675. Begin on the most convenient element.

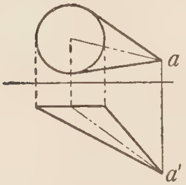

Fig. 668

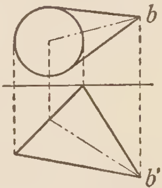

Fig. 669

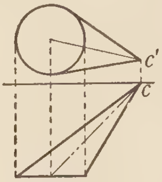

Fig. 670

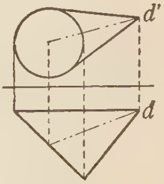

Fig. 671

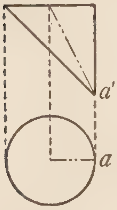

Fig. 672

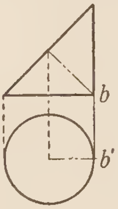

Fig. 673

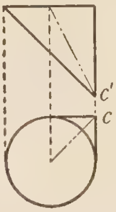

Fig. 674

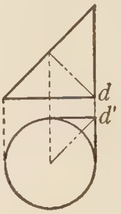

Fig. 675

375. Develop the half pattern, as determined by a plane of symmetry, of any assigned truncated cone shown in Figs. 676 to 679.

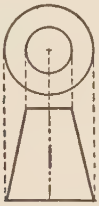
Fig. 676

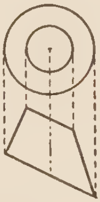
Fig. 677

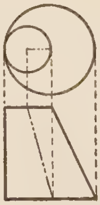
Fig. 678

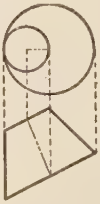
Fig. 679

376. Develop a fourth or a half pattern, as determined by planes of symmetry, of any assigned transition piece shown in Figs. 680 to 685.

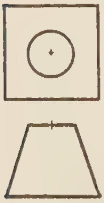
Fig. 680

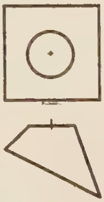
Fig. 681

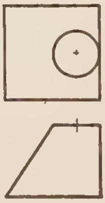
Fig. 682

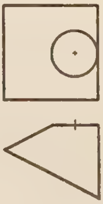
Fig. 683

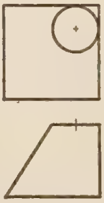
Fig. 684

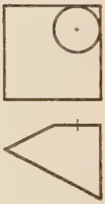
Fig. 685

Article 107. The Development (Pattern) of a Cylinder

377. Develop the half pattern, as determined by a plane of symmetry, of any assigned cylinder shown in Figs. 686 to 697.

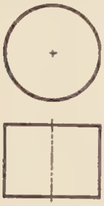
FIG. 686

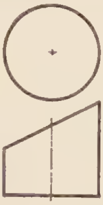
FIG. 687

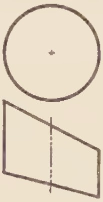
FIG. 688

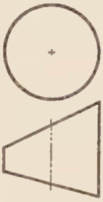
FIG. 689

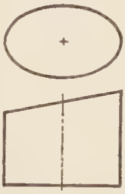
FIG. 690

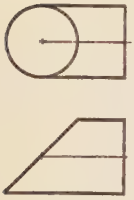
FIG. 691

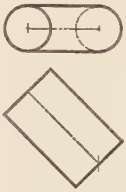
FIG. 692

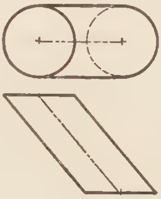
FIG. 693

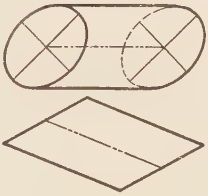
FIG. 694

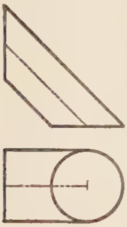
FIG. 695

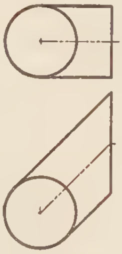
FIG. 696

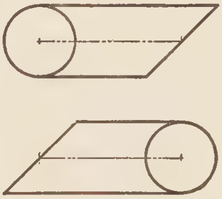
FIG. 697

Article 111. A Line making Specified Angles Θ, Φ, or Π with Two of the Coördinate Planes

378. Draw a line through any assigned point shown in Figs. 1 to 15, page 265, so that any two of its coördinate angles Θ, Φ, and Π shall have specified values.

Article 112. A Line in a Given Plane and making a Specified Angle with One of the Coördinate Planes

379. Draw a line in any assigned plane shown in Figs. 133 to 157, page 271, making a specified angle Θ, Φ, or Π with one of the coördinate planes.

Article 113. The Coördinate Angles Θ, Φ, and Π of a Given Plane

380. Find the angle that any assigned plane shown in Figs. 133 to 157, page 271, makes with each coördinate plane, that is, the angles Θ, Φ, and Π.

Article 115. A Plane through a Given Point, making a Specified Angle Θ, Φ, or Π with the Corresponding Coördinate Plane

381. Pass a plane through any assigned point shown in Figs. 1 to 15, page 265, so that any one of its coördinate angles Θ, Φ, or Π shall have a specified value.

Article 116. A Plane through a Given Point, making Specified Angles Θ, Φ, or Π with Two of the Coördinate Planes

382. Pass a plane through any assigned point shown in Figs. 1 to 15, page 265, so that any two of its coördinate angles Θ, Φ, and Π shall have specified values.

Article 117. A Plane through a Given Line and making a Specified Angle Θ, Φ, or Π with the Corresponding Coördinate Plane

383. Pass a plane through any assigned line shown in Figs. 55 to 79, page 267, so that any one of its coördinate angles Θ, Φ, or Π shall have a specified value.

384. The line AB, Fig. 698, is one side of a square ABCD whose plane is inclined 60° to the V plane. Draw the projections of any one of the four possible squares.

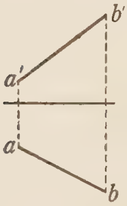

FIG. 698

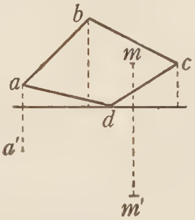

FIG. 699

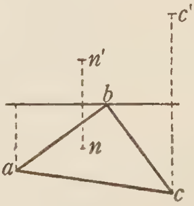

FIG. 700

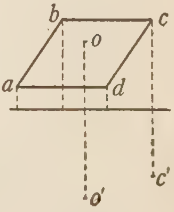

FIG. 701

385. The point M, Fig. 699, is in the plane of the quadrilateral ABCD, which is inclined 60° to the H plane. Find the V projection of ABCD.

386. The point N, Fig. 700, is in the plane of the triangle ABC, which is inclined 60° to the V plane. Find the V projection of ABC.

387. The point O, Fig. 701, is in the plane of the quadrilateral ABCD, which is inclined 45° to the P plane. Find the V projection of ABCD.

Articles 118 and 119. The Plane Tangent to a Cone or a Cylinder through a Point on Either Surface

388. Pass a plane tangent to any assigned cone or cylinder shown in Figs. 702 to 719 and through the point O on the surface.

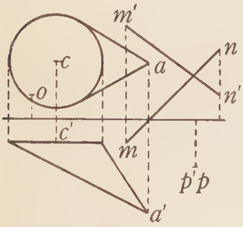

FIG. 702

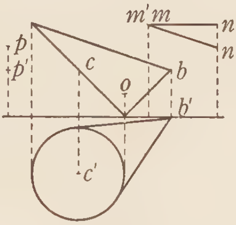

FIG. 703

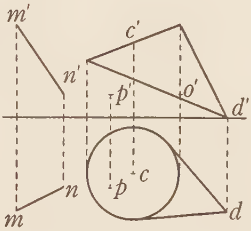

FIG. 704

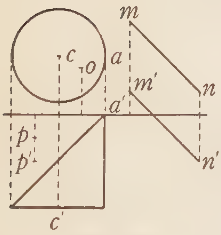

FIG. 705

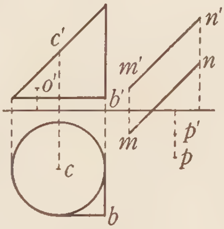

FIG. 706

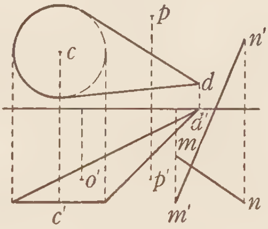

FIG. 707

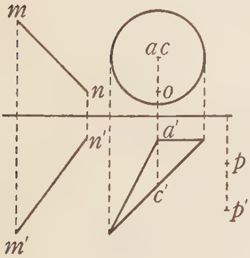

FIG. 708

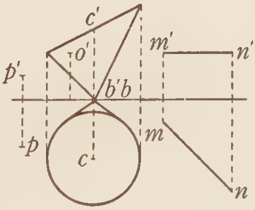

FIG. 709

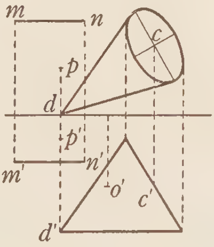

FIG. 710

Article 120. A Plane Tangent to a Cone or a Cylinder through a Point outside Either Surface

389. Pass a plane tangent to any assigned cone or cylinder shown in Figs. 702 to 719 and through the point P outside the surface.

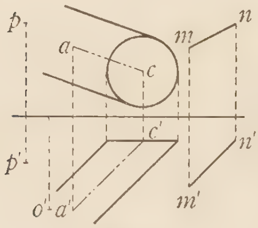

FIG. 711

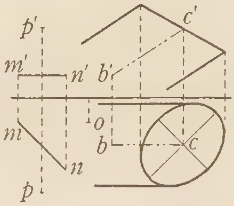

FIG. 712

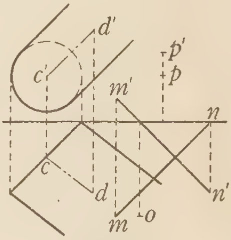

FIG. 713

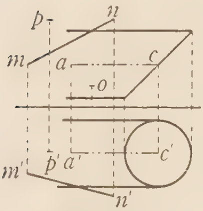

FIG. 714

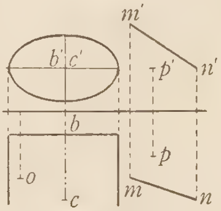

FIG. 715

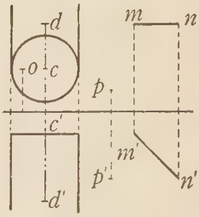

FIG. 716

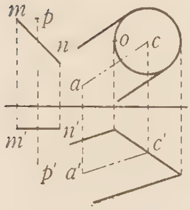

FIG. 717

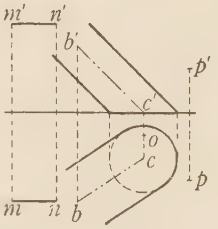

FIG. 718

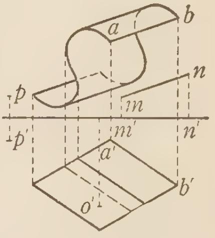

FIG. 719

Article 121. A Plane Tangent to a Cone and Parallel to a Line

390. Pass a plane tangent to any assigned cone shown in Figs. 702 to 710 and parallel to the given line MN.

Article 122. A Plane Tangent to a Cylinder and Parallel to a Line

391. Pass a plane tangent to any assigned cylinder shown in Figs. 711 to 719 and parallel to the given line MN.

Article 125. A Plane Tangent to a Surface of Revolution through a Point on the Surface

392. Pass a plane tangent to any assigned surface of revolution shown in Figs. 720 to 726 and through the point O on the surface.

Articles 126 and 127. A Plane Tangent to a Surface of Revolution through a Point outside the Surface

393. Pass a plane tangent to any assigned surface of revolution shown in Figs. 720 to 726 and through the given point P outside the surface.

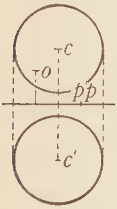

FIG. 720

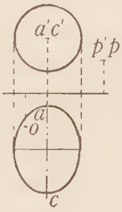

FIG. 721

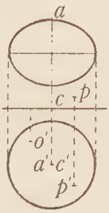

FIG. 722

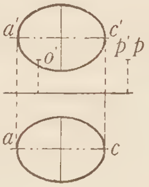

FIG. 723

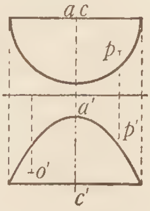

FIG. 724

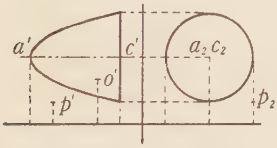

FIG. 725

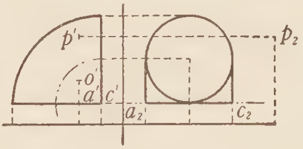

FIG. 726

Article 128. A Line or a Plane making a Specified Angle with a Given Plane

394. Draw a line OM through the given point O and making a specified angle X with the given plane in any assigned set shown in Figs. 263 to 282, page 277.

395. Draw a line OM through any assumed point O and making a specified angle X with any assigned polygon shown in Figs. 323 to 338, page 280.

396. Pass a plane through the given point O and making a specified angle X with the given plane in any assigned set shown in Figs. 263 to 282, page 277.

397. Pass a plane through any assumed point O and making a specified angle X with any assigned polygon shown in Figs. 323 to 338, page 280.

Articles 130 to 135. Isometric Projection or Drawing

398. Make an isometric drawing of any assigned object shown in Figs. 727 to 746. Choose a suitable scale.

Articles 136 to 142. Oblique Projection or Drawing

399. Make an oblique drawing of any assigned object shown in Figs. 727 to 746. Choose a suitable scale.

Articles 143 to 151. Perspective Drawing

NOTE. The position of the point of sight, S, in all problems under Perspective Drawing is specified with reference to some chosen point A on the object. For example, S(– 2, 4, 9) means that the point S is 2″ to the left of point A, 4″ above A, and 9″ before A. Positive and negative values of these coördinates have the same meaning as in the coördinate system explained in Article 8.

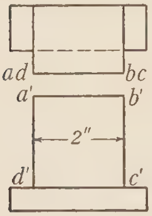

FIG. 727

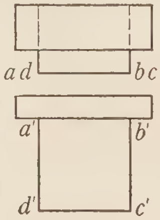

FIG. 728

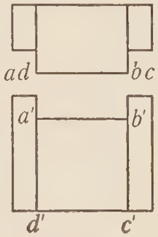

FIG. 729

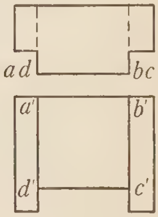

FIG. 730

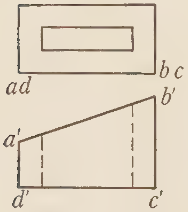

FIG. 731

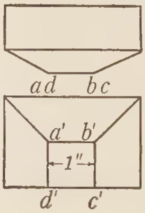

FIG. 732

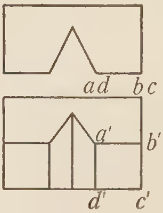

FIG. 733

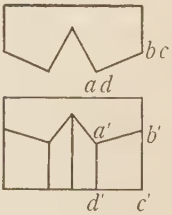

FIG. 734

400. Place any assigned object shown in Figs. 727 to 734 with the face ABCD in an assumed picture plane, the edge BC being vertical. Draw the perspective of the object by the visual-ray method, taking the point of sight S at ($2\frac{1}{2}$, 1, 6) from point A, full scale.

401. Construct the perspective of Problem 400 by the vanishing-point method.

402. Place any assigned object shown in Figs. 727 to 734 with the edge BC in a vertical position in an assumed picture plane, and the face ABCD inclined 30° to the picture plane. Construct the perspective of the object by the visual-ray method, taking the point of sight S at ($2\frac{1}{2}$, 1, 6) from point A, full scale.

403. Construct the perspective of Problem 402 by the vanishing-point method.

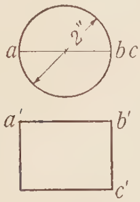

FIG. 735

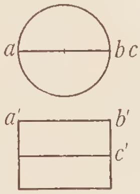

FIG. 736

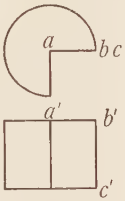

FIG. 737

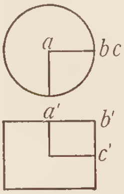

FIG. 738

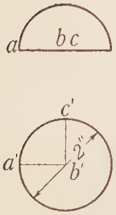

FIG. 739

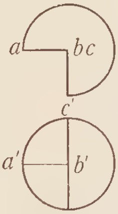

FIG. 740

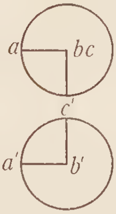

FIG. 741

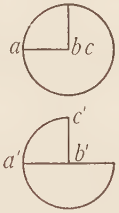

FIG. 742

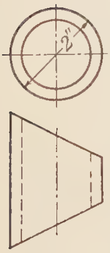

FIG. 743

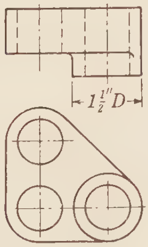

FIG. 744

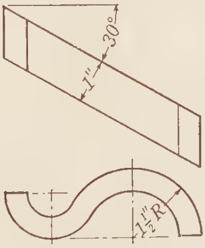

FIG. 745

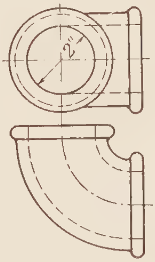

FIG. 746

404. Place any assigned object shown in Figs. 735 to 742 with the line BC vertical and the plane of ABC $1\frac{1}{4}''$ behind an assumed picture plane. Draw the perspective of the object by the visual-ray method, taking the point of sight S at (3, 2, 8) from point A, full scale.

405. Construct the perspective of Problem 404 by the vanishing-point method.

Articles 152 to 158. Shades and Shadows

NOTE. It is the practice in obtaining the shadows of lines, surfaces, and solids, on the H and V planes, to consider only that portion of the V plane above the H plane and that portion of the H plane in front of the V plane as available. If any other plane is set up, it is considered opaque, so that the light rays do not pass through it, thus preventing the object from casting a shadow on certain portions of the H and V planes.

406. Find the shadow of any assigned line shown in Figs. 747 to 751 upon the H and V planes.

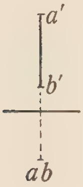

FIG. 747

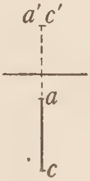

FIG. 748

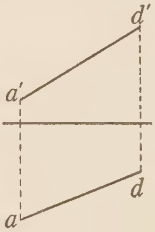

FIG. 749

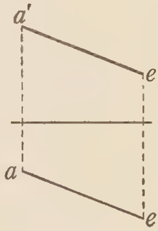

FIG. 750

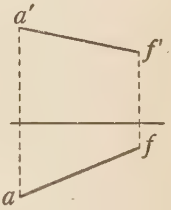

FIG. 751

407. Find the shadow of any assigned object shown in Figs. 752 to 766 upon the V plane.

408. Find (*a*) the shadow of the line AB, Fig. 767, upon the prism MN, and (*b*) the shadow of AB and MN upon the H and V planes.

409. Find the shadow of the square ABCD, Fig. 768, upon the plane R.

410. Find (*a*) the shadow of the cube AC, Fig. 769, upon the plane S, and (*b*) the shadow of the cube upon the H plane in front of the plane S.

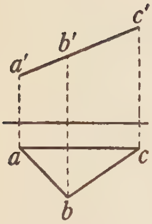

FIG. 752

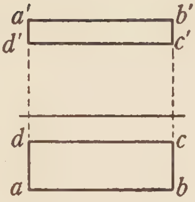

FIG. 753

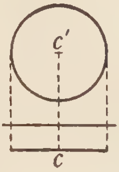

FIG. 754

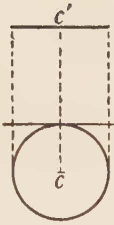

FIG. 755

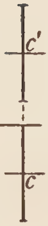

FIG. 756

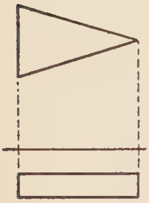

FIG. 757

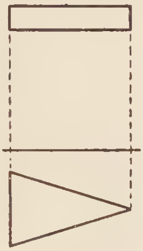

FIG. 758

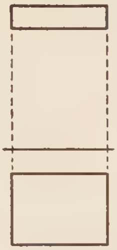

FIG. 759

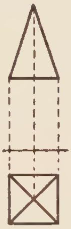

FIG. 760

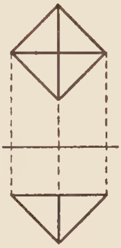

FIG. 761

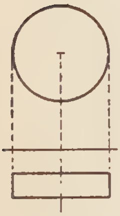

FIG. 762

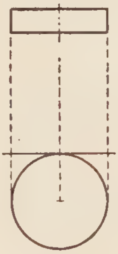

FIG. 763

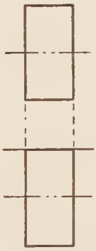

FIG. 764

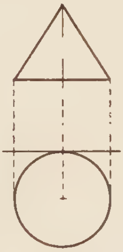

FIG. 765

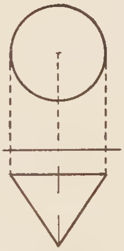

FIG. 766

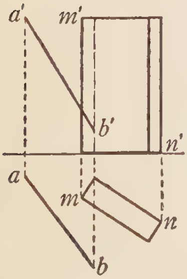

FIG. 767

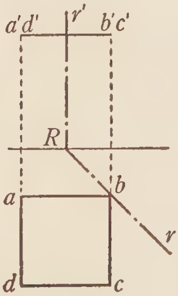

FIG. 768

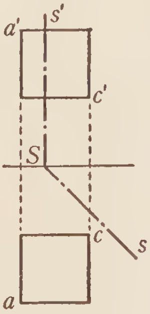

FIG. 769

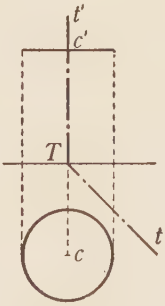

FIG. 770

411. Find the shadow of the circle whose center is C, Fig. 770, upon the plane T.

412. Find the shadow of the half cylinder of Fig. 771 upon the plane R.

413. Find the shadow of the cylinder of Fig. 772 upon the plane S.

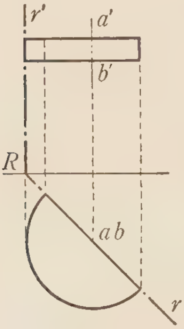

Fig. 771

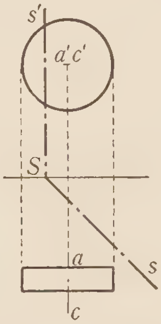

Fig. 772

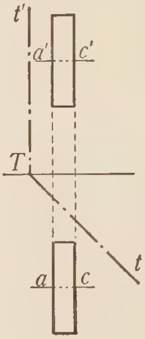

Fig. 773

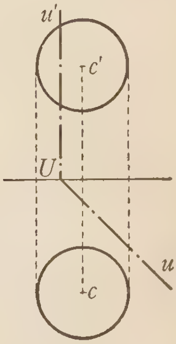

Fig. 774

414. Find the shadow of the cylinder of Fig. 773 upon the plane T.

415. Find the shadow of the sphere of Fig. 774 upon the plane U.

416. Find (*a*) the shadow of the line AB, Fig. 775, on the V plane and on the cylinder; (*b*) the shadow of the cylinder on the V plane; and (*c*) the shade of the cylinder.

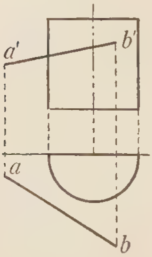

Fig. 775

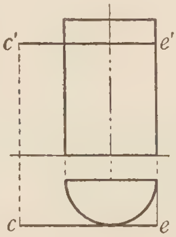

Fig. 776

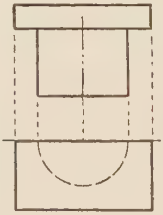

Fig. 777

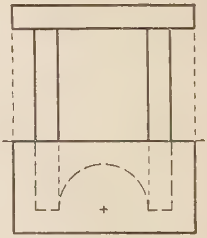

Fig. 778

417. Find (*a*) the shadow of the line CE, Fig. 776, upon the V plane and upon the cylinder; (*b*) the shadow of the cylinder upon the V plane; and (*c*) the shade of the cylinder.

418. Find (*a*) the shadow of the rectangular cap on the circular shaft of the object shown in Fig. 777; (*b*) the shadow of the object upon the V plane; and (*c*) the shade of the object.

419. Find (*a*) the shadow of the rectangular cap on the shaft of the object shown in Fig. 778; (*b*) the shadow of the object upon the V plane; and (*c*) the shade of the object.

420. Find (*a*) the shadow of the line AB, Fig. 779, on the molding shown; (*b*) the shadow of AB and the molding upon the V plane; and (*c*) all shades of the molding.

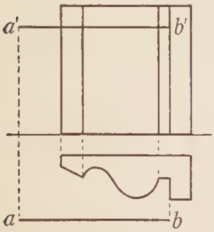

FIG. 779

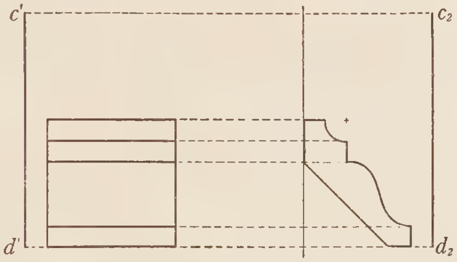

FIG. 780

421. Find (*a*) the shadow of the line CD, Fig. 780, upon the molding shown, and (*b*) the shadow of CD and the molding upon the V plane.

422. Find (*a*) the shadow of the line AB, Fig. 781, upon the object shown and upon the V plane; (*b*) the shadow of the object upon the V plane; and (*c*) the shade of the object.

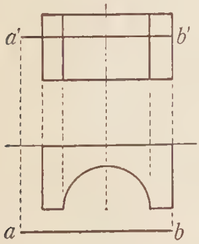

FIG. 781

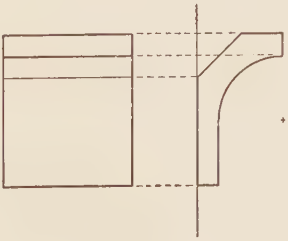

FIG. 782

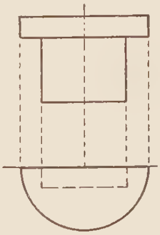

FIG. 783

423. Find (*a*) the shade of the molding shown in Fig. 782, and (*b*) the shadow of the molding upon the V plane.

424. Find (*a*) the shadow of the circular cap on the rectangular shaft of the object shown in Fig. 783; (*b*) the shadow of the object upon the V plane; and (*c*) the shade of the object.

425. Find (*a*) the shadow of the circular cap on the circular shaft of the object shown in Fig. 784; (*b*) the shadows of the object upon the V and H planes; and (*c*) the shade of the object.

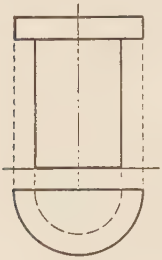
FIG. 784

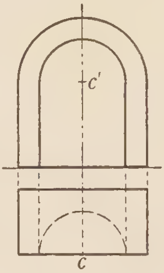

FIG. 785

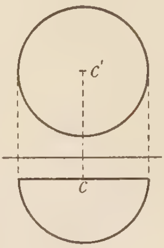

FIG. 786

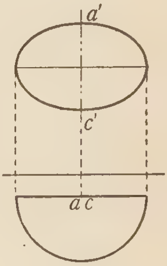

FIG. 787

426. Find (*a*) the shadow of the upper front edge of the niche shown in Fig. 785; (*b*) the shadow of the object on the V and H planes; and (*c*) the shades of the object.

427. Find the shade line of the hemisphere shown in Fig. 786.

428. Find the shade line of the hemispheroid of Fig. 787.

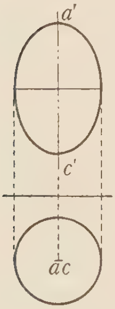

FIG. 788

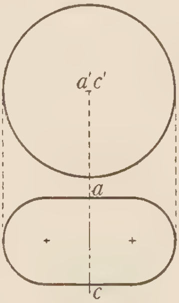

FIG. 789

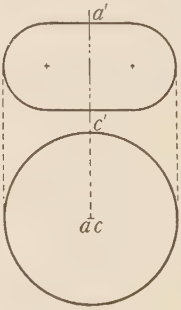

FIG. 790

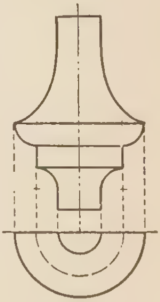
FIG. 791

429. Find the shade line of the prolate spheroid of Fig. 788.

430. Find the shade line of the torus of Fig. 789.

431. Find the shade line of the torus of Fig. 790.

432. Find (*a*) the shades of the object shown in Fig. 791, and (*b*) the shadow of the object upon the V plane.

433. A 2-inch sphere is placed with its center at the point $C(x, 2, 2)$. Find its shadow on the H and V planes.

434. A 1″ × 2″ prolate spheroid is placed with its axis perpendicular to the H plane, with its center at the point $C(x, 2, 2)$. Find its shadow on the H and V planes.

435. A 1″ × 2″ oblate spheroid is placed with its axis perpendicular to the H plane, with its center at the point $C(x, 2, 2)$. Find its shadow on the H and V planes.

436. An annular torus is generated by a 1-inch circle revolving about an axis perpendicular to the H plane and 1″ from the center of the circle. The center of the torus is at the point $C(x, 2, 4)$. Find the shadow of the torus on the H plane.

437. The torus of Problem 436 is placed with its center at the point $C(x, 4, 2)$. Find its shadow upon the V plane.

438. A right circular cone whose elements are inclined 45° to the axis is placed with the axis perpendicular to the H plane and the base downwards. Find how much of the surface is in shade.

439. If the axis of the cone of Problem 438 is perpendicular to the V plane, the base forward, find how much of the surface is in shade.

440. If the axis of the cone of Problem 438 is perpendicular to the P plane, the base to the right, find how much of the surface is in shade.

441. Draw the cone of Problem 438, and find its shadow upon the V plane.

442. Draw the cone of Problem 439, and find its shadow upon the V plane.

443. Draw the cone of Problem 440, and find its shadow upon the V plane.

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